MATH 4707, HOMEWORK II, SOLUTIONS TO GRADED PROBLEMS

February 25, 2010

INSTRUCTOR: Anar Akhmedov

(1) 4.3.6

Solution: Let a_n denote the number of all such subsets of $\{1, 2, \cdots, n\}$. We will find a recurrence relation for a_n. There are two types of subsets, the subsets that contain 1 and the subsets that do not contain 1. If the subset do not contain 1, then there are F_{n+1} possible subsets of $\{2, 3, \cdots, n\}$ without consecutive elements (see Problem 4.1.3 page 67 and the solution on page 260). If the subset do contain 1, then it can not contain 2 and n. There are F_{n-1} possible subsets of $\{3, 4, \cdots, n-2\}$ without consecutive elements.

Thus, we get the following recurrence: $a_n = F_{n+1} + F_{n-1}$.

(2) 4.3.8

Solution We give an inductive proof. By experiment, we check that $d_n = 2^n - F_n > 0$ for small n. For example, $d_1 = 2^1 - F_1 = 1 > 0$. Assume that $d_{n-1} > 0$ holds for all $n \geq 2$, and we will show that $d_n > 0$ holds. $d_n = 2^n - F_n = 2^n - F_{n-1} - F_{n-2} \geq 2^n - 2F_{n-1} = 2(2^{n-1} - F_{n-1}) = 2d_{n-1} > 0$.

(3) 4.3.9

Solution:

a) We give an inductive proof. It is easy to check that $F_2+F_4+F_6+\cdots+F_{2n} = F_{2n+1}-1$ holds for small n. Assume that $F_2+F_4+F_6+\cdots+F_{2k} = F_{2k+1}-1$ for all $k \leq n-1$, we will show that it holds for $k = n$. $F_2+F_4+F_6+\cdots+F_{2n} = (F_{2n-1}-1)+F_{2n} = F_{2n+1}-1$.

b) $F_{n+1}^2 - F_n^2 = (F_{n+1} - F_n)(F_{n+1} + F_n) = F_{n-1}F_{n+2}$

We will use Binet's formula for the Fibonacci numbers and Binomial Theorem to prove c) and d). We have $F_n = \frac{1}{a-b}(a^n - b^n)$, where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Since a and b are solutions of the equation $x^2 - x - 1 = 0$, we have the following identities: $a^2 = a+1$, $b^2 = b+1$, $a+b = 1$, and $ab = -1$.

c) $\binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \cdots + \binom{n}{n}F_n = \binom{n}{0}\frac{1}{a-b}(a^0 - b^0) + \binom{n}{1}\frac{1}{a-b}(a^1 - b^1) + \binom{n}{2}\frac{1}{a-b}(a^2 - b^2) + \cdots + \binom{n}{n}\frac{1}{a-b}(a^n - b^n) = \frac{1}{a-b}((\binom{n}{0}a^0 + \binom{n}{1}a^1 + \binom{n}{2}a^2 + \cdots + \binom{n}{n}a^n) - (\binom{n}{0}b^0 + \binom{n}{1}b^1 + \binom{n}{2}b^2 + \cdots + \binom{n}{n}b^n)) = \frac{1}{a-b}((a+1)^n - (b+1)^n) = \frac{1}{a-b}((a^2)^n - (b^2)^n) = \frac{1}{a-b}(a^{2n} - b^{2n}) = F_{2n}$

d) $\binom{n}{0}F_1 + \binom{n}{1}F_2 + \binom{n}{2}F_3 + \cdots + \binom{n}{n}F_{n+1} = \binom{n}{0}\frac{1}{a-b}(a^1 - b^1) + \binom{n}{1}\frac{1}{a-b}(a^2 - b^2) + \binom{n}{2}\frac{1}{a-b}(a^3 - b^3) + \cdots + \binom{n}{n}\frac{1}{a-b}(a^{n+1} - b^{n+1}) = \frac{1}{a-b}(a(\binom{n}{0}a^0 + \binom{n}{1}a^1 + \binom{n}{2}a^2 + \cdots + \binom{n}{n}a^n) - b(\binom{n}{0}b^0 + \binom{n}{1}b^1 + \binom{n}{2}b^2 + \cdots + \binom{n}{n}b^n)) = \frac{1}{a-b}(a(a+1)^n - b(b+1)^n) = \frac{1}{a-b}(a(a^2)^n - b(b^2)^n) = \frac{1}{a-b}(a^{2n+1} - b^{2n+1}) = F_{2n+1}$

(4) 5.4.1

Solution: Let A denote the event that the sum of the points is 8 if we throw a die twice. $A = \{(4,4), (5,3), (3,5), (6,2), (2,6)\}$. Since the sample space S has 36 elements, we have $P(A) = |A|/|S| = 5/36$.

(5) 5.4.3

Solution:

Since A and B are independent events, we have $P(A \cap B) = P(A)P(B)$. Thus, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B)$.