MATH 4707, HOMEWORK III, SOLUTIONS TO GRADED PROBLEMS

April 10, 2010

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(1) 6.10.6

Solution: Since k divides n, we have $n = ks$ for some integer s. $a^n - 1 = a^{ks} - 1 = (a^k)^s - 1 = (a^k - 1)((a^k)^{s-1} + (a^k)^{s-2} + (a^k)^{s-3} + \cdots (a^k)^2 + a^k + 1)$. Thus, $a^k - 1$ divides $a^n - 1$.

Now suppose that $a^k - 1$ divides $a^n - 1$. Let $n = ks + r$, where r $(0 \le r \le k-1)$ is a remainder when we divide n by k. $a^n - 1 = a^{ks}a^r - 1 = ((a^k)^s - 1)a^r + (a^r - 1)$. By the argument above, $a^k - 1$ divides $a^{ks} - 1$. Thus, $a^k - 1$ must divide $a^r - 1$. Since $r < k$, we have $r = 0$.

(2) 6.10.19

Solution: We prove the given statement by contradiction. Let's assume that $\sqrt[3]{5} = a/b$, where a and b are relatively prime integers. By cubing both sides of the above equation, we have $5b^3 = a^3$. Since 5 divides the left hand side of the last equation, we get 5 divides a. Let $a = 5c$ for some integer c. We have $5b^3 = (5c)^3 = 125c^3$, which simplifies to $b^3 = 25c^3$.Thus, $b = 5d$ for some integer d. So, 5 divides both a and b. This contradicts to the fact that a and b are realtively prime.

(3) 6.10.22

Solution: Let's divide the given set $X = \{1, 2, \ldots, 2n\}$ into the following n pairs: $p_1 = \{1, 2\}, p_2 = \{2, 4\}, p_3 = \{3, 6\}, \cdots, p_i = \{i, 2i\}, \cdots, p_n = \{n, 2n\}$. By Pigeonhole Principle, there are two numbers among these $n + 1$ numbers that belong to same p_k. Thus, we get two numbers such that one divides the other.

(4) 7.3.5

Solution:

a) No. The graph (with 7 nodes) can't have both a vertex of degree 6 and a vertex of degree 0.

b) No. In any graph, the number of nodes with odd degree must be even.

(5) 7.3.9

Solution: If the given graph G is connected then we are done. Otherwise, G is the union of its connected components $G_1, G_2, \cdots, G_m$, where $m \ge 2$. If the vertices u and v belong to the different components G_i and G_j of G, then they are connected in $\bar{G}$. If the vertices u and w belong to the same component G_i of G, then they are connected by a path $u \to v \to w$ for some vertex v that belongs to G_j, where $j \ne i$. This shows that $\bar{G}$ is connected.

(6) 7.3.13

Solution: We will give an inductive proof based on m. Note that if $m = 2$, then we have either two people who are strangers or all these $n + 1$ people know each other. Suppose that our claim is true for $k = m \ge 2$, and we will prove it for $k = m + 1$. Let X be any person among these $mn + 1$ people. If X knows at least n others, then we are done. Otherwise, X knows at most $n - 1$ others. There are at least $n(m - 1) + 1$ people who don't know X. By induction, among these $m(n - 1) + 1$ people there are either one person who knows n others or m people $Y_1, Y_2, \cdots Y_m$ who don't know each other. If the first case holds, then we are done. If not, then we have $m + 1$ people $X, Y_1, Y_2, \cdots, Y_m$ who are strangers.