MATH 4707, HOMEWORK IV, SOLUTIONS TO GRADED PROBLEMS

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INSTRUCTOR: Anar Akhmedov

(1) 8.5.3

Solution: If G' is a connected graph with n nodes, then it has at least $n-1$ edges. Let assume that the given graph has k connected components $G_1, G_2 \cdots, G_k$. We will denote by $n_1, n_2, \cdots, n_k$ the number of vertices in the corresponding components. Then the graph G has $n = n_1 + \cdots n_k$ vertices. Since the graph G has k connected components, it follows from the above fact that it has at least $n-k$ edges. Hence, the number of edges $m \geq n-k$. This inequality implies that $k \geq n-m$.

(2) 8.5.4

Solution: Let G be any tree with n vertices. Suppose that a vertex v_1 in G has degree d, and there are k vertices, $v_2, \cdots v_{k+1}$, of degree 1. If $d=2$, then by Theorem 8.2.1 we have at least 2 leaves, so we are done. Next, we consider the case $d \geq 3$. Notice that the graph G has $n-k-1$ other vertices $v_{k+2}, \cdots v_n$ of degree at least 2. By Handshaking Lemma and Theorem 8.2.3, we have $2(n-1) = 2e_G = deg(v_1) + deg(v_2) + \cdots + deg(v_n) = d + k + deg(v_{k+2}) + \cdots deg(v_n) \geq d + k + 2(n-k-1)$, which implies that $d \leq k$.

(3) 10.4.9

Solution: Let $G = (X, Y)$ be the given bipartite graph. We will apply Theorem 10.3.1 (Hall's Theorem). Let S be any subset of X. If $|S| \geq m/2$, then $|N_G(S)| \geq |S|$. If $|S| > m/2$, we will show that $|N_G(S)| \geq |S|$ as well.

Let's assume that $|N_G(S)| < |S|$. Let y be any vertex in $Y - N_G(S)$. Since y can only be connected with the vertices in $X - S$, the degree of y can't be more than the number of elements in $X - S$. Since $|S| > m/2$, the degree of y will be less than $m/2$. This is a contradiction to the given statment. Thus, if $|S| > m/2$, $|N_G(S)| \geq |S|$. Now we are ready to apply Hall's Theorem.

(4) 10.4.15

Solution: Note that we have $1-1$ correspondence between the perfect matchings of the ladder graph and the tilings of a $2 \times n$ grid. Identify vertical edge with a vertical domino, and horizontal edge with a horizontal domino. Thus, the number of perfect matchings of the given graph is F_n, nth Fibonacci number.