# MATH 4707 MIDTERM I

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Show all of your work. No credit will be given for an answer without some work or explanation.

| Problem | Points |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
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| 6 | |
| Total (150 points) | |

1. (26 points) Prove in two ways that $\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}=\binom{n+m}{k}$.

   *Combinatorial Proof:* Let $X=\{1,2,3,\ldots,m+n-1,m+n\}$. We will write $X$ as the disjoint union of the subsets $A$ and $B$, where $A=\{1,2,3,\ldots,n-1,n\}$ and $B=\{n+1,n+2,\ldots,n+m-1,n+m\}$. Note that $|A|=n$ and $|B|=m$. The binomial coefficient $\binom{n+m}{k}$ is the number of $k$ subsets of the set $X$, which gives the right hand side of the given identity. Each such $k$-subset contains $i$ elements from $A$ (for some $0\le i\le n$) and the rest $k-i$ elements from $B$. The summation for all $i$ gives the left hand side of the identity.

   *Algebraic Proof:* By the Binomial Theorem, we have $(1+x)^{m+n}=\sum_{k=0}^{n+m}\binom{n+m}{k}x^k$. Also, we have $(1+x)^{m+n}=(1+x)^n(1+x)^m=(\sum_{i=0}^{n}\binom{n}{i}x^i)(\sum_{j=0}^{m}\binom{m}{j}x^j)=\sum_{k=0}^{n+m}(\sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-1})x^k$. By comparing the coefficents of $x^k$, we obtain the given identity.

2. (24 points) Let $P_i=(x_i,y_i,z_i)$, $i=1,2,3,\cdots,9$ be a set of nine distinct points with integer coordinates in $\mathbf{R^3}$. Show that the midpoint of at least one pair of these points has integer coordinates.

   *Solution:* The midpoint of two points $P_i=(x_i,y_i,z_i)$ and $P_j=(x_j,y_j,z_j)$ with integer coordinates has the integer coordinates if and only if $x_i$ and $x_j$, $y_i$ and $y_j$, and $z_i$ and $z_j$ have the same parity.

   The followings are the eight possible parities of a point with integer coordinates: *(EVEN,EVEN,EVEN), (EVEN,EVEN,ODD), (EVEN,ODD,EVEN), (EVEN,ODD,ODD), (ODD,EVEN,EVEN), (ODD,EVEN,ODD), (ODD,ODD,EVEN), and (ODD,ODD,ODD).* By the Pigeonhole Principle, among the given nine points, there will be two points with the same parity. The midpoint of such two points has the integer coordinates.

3. (24 points) Let $X=\{1,2,\ldots,1000\}$. How many numbers in $X$ are multiples of 2, 3, or 5?

   *Solution:* We will use the Inclusion-Exclusion Principle: Let $A$, $B$, and $C$ be the set of positive integers in $X$ that are divisible by 2, 3, and 5, respectively. We need to compute $|A\cup B\cup C|$. By the inclusion-exclusion formula, we have $|A\cup B\cup C|=|A|+|B|+|C|-|A\cap B|-|B\cap C|-|A\cap C|+|A\cap B\cap C|$. $|A|=10000/2=500$, $B=[1000/3]=333$, $C=1000/5=200$, $|A\cap B|=[1000/6]=166$, $|B\cap C|=[1000/15]=66$, $|A\cap C|=1000/10=10$, and $|A\cap B\cap C|=[1000/30]=33$. By the formula above, we have $|A\cup B\cup C|=500+333+200-(166+66+100)+33=734$.

4. (26 points) Which of the Fibonacci numbers $F_{100}$, $F_{101}$, $F_{102}$, $F_{103}$, $F_{104}$, and $F_{105}$ are

   a) divisible by 2

   b) divisible by 3

   *Solution:* We'll use the following fact that we proved in class. Please also see Problem 4.2.8 (textbook, page 71) and it's solution (page 262). If $n$ is a multiple of $k$, then $F_n$ is a multiple of $F_k$. Since $2=F_3$ and $3=F_4$, the above result implies that a) $F_{102},F_{105}$ are divisible by $F_3=2$, and b) $F_{100}$, $F_{104}$ are divisible by $F_4=3$.

5. (24 points) A probability space contains two events $A$ and $B$ such that $P(A)=1/5$, $P(B)=2/5$, and $P(A\cap B)=1/10$. Find the followings

   a) $P(A^c)$

   $P(A^c)=1-P(A)=1-1/5=4/5$.

   b) $P(A\cup B)$

   $P(A\cup B)=P(A)+P(B)-P(A\cap B)=1/5+2/5-1/10=1/2$.

   c) $P(A^c\cup B^c)$

   $P(A^c\cup B^c)=P((A\cap B)^c)=1-P(A\cap B)=1-1/10=9/10$.

   d) Are $A$ and $B$ independent?

   Since $P(A\cap B)=1/10\neq P(A)P(B)=2/25$, the events $A$ and $B$ are not independent.

6. (26 points) Prove the following two identities

a) $\sum_{k=0}^{n} \binom{n}{k} 2^k F_k = F_{3n}$

b) $\sum_{k=0}^{n} \binom{n}{k} 2^k L_k = L_{3n}$

*Solution:* We will use Binet's formula for the Fibonacci numbers, the corresponding formula for the Lucas numbers, and Binomial Theorem to prove a) and b). We have $F_n = \frac{1}{a-b}(a^n - b^n)$ and $L_n = a^n + b^n$, where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$. Since $a$ and $b$ are solutions of the equation $x^2 - x - 1 = 0$, we have the following identities hold: $a^2 = a+1$, $b^2 = b+1$. Using these identities, we have $a^3 = a^2 a = (a+1)a = a^2 + a = 2a+1$, and $b^3 = b^2 b = (b+1)b = b^2 + b = 2b+1$.

a) $\binom{n}{0} 2^0 F_0 + \binom{n}{1} 2^1 F_1 + \binom{n}{2} 2^2 F_2 + \cdots + \binom{n}{n} 2^n F_n = \binom{n}{0} 2^0 \frac{1}{a-b}(a^0 - b^0) + \binom{n}{1} 2^1 \frac{1}{a-b}(a^1 - b^1) + \binom{n}{2} 2^2 \frac{1}{a-b}(a^2 - b^2) + \cdots + \binom{n}{n} 2^n \frac{1}{a-b}(a^n - b^n) = \frac{1}{a-b}((\binom{n}{0} 2^0 a^0 + \binom{n}{1} 2^1 a^1 + \binom{n}{2} 2^2 a^2 + \cdots + \binom{n}{n} 2^n a^n) - (\binom{n}{0} 2^0 b^0 + \binom{n}{1} 2^1 b^1 + \binom{n}{2} 2^2 b^2 + \cdots + \binom{n}{n} 2^n b^n)) = \frac{1}{a-b}((2a+1)^n - (2b+1)^n) = \frac{1}{a-b}((a^3)^n - (b^3)^n) = \frac{1}{a-b}(a^{3n} - b^{3n}) = F_{3n}$.

b) $\binom{n}{0} 2^0 L_0 + \binom{n}{1} 2^1 L_1 + \binom{n}{2} 2^2 L_2 + \cdots + \binom{n}{n} 2^n L_n = \binom{n}{0} 2^0 (a^0 + b^0) + \binom{n}{1} 2^1 (a^1 + b^1) + \binom{n}{2} 2^2 (a^2 + b^2) + \cdots + \binom{n}{n} 2^n (a^n + b^n) = ((\binom{n}{0} 2^0 a^0 + \binom{n}{1} 2^1 a^1 + \binom{n}{2} 2^2 a^2 + \cdots + \binom{n}{n} 2^n a^n) + (\binom{n}{0} 2^0 b^0 + \binom{n}{1} 2^1 b^1 + \binom{n}{2} 2^2 b^2 + \cdots + \binom{n}{n} 2^n b^n)) = ((2a+1)^n + (2b+1)^n) = ((a^3)^n + (b^3)^n) = a^{3n} + b^{3n} = L_{3n}$.