On the finiteness of the class number for an algebraic number field K...

We know that any fractional ideal $\mathfrak{A}$ can be written uniquely in the form

$\mathfrak{A} = (\mathfrak{p}_1 ... \mathfrak{p}_s)/(\mathfrak{p}_1^{'} ... \mathfrak{p}_r^{'})$

where the $\mathfrak{p}'s$ are primes in $\mathfrak{O}_K$ and no $\mathfrak{p}_i$ is a $\mathfrak{p}_j^{'}$. One can always express a fractional ideal $\mathfrak{A} = \mathfrak{b}/\mathfrak{c}$. where $\mathfrak{b}$ and $\mathfrak{c}$ are two integral ideals. Two fractional ideals $\mathfrak{A}$ and $\mathfrak{B}$ are defined to be equivalent if there exist α and β in the ring of integers so that $(\alpha)\mathfrak{A} = (\beta)\mathfrak{B}$. It is clear to see that this defines an equivalence relation on fractional ideals.

There is an important result asserting the existence of a constant C_K such that every ideal $\mathfrak{a}$ in the ring of integers is equivalent to another ideal $\mathfrak{b}$ in the ring of integers such that $N(\mathfrak{b}) \leq C_K$.

In order to establish the finiteness of the class number,we must show that each equivalence class of ideals has an integral ideal representative. Indeed, suppose that $\mathfrak{A}$ is a fractional ideal in K. Let $\mathfrak{A} = \mathfrak{b}/\mathfrak{c}$ with $\mathfrak{b}$ and $\mathfrak{c}$ in $\mathfrak{O}_K$. We know(avoiding a bit of work here) that $\mathfrak{c} \cap \mathbb{Z} \neq 0$. Therefore there is a nonzero t in $\mathfrak{c} \cap \mathbb{Z}$. Therefore, $\mathfrak{c} \supseteq (t)$, so $\mathfrak{c}$ divides (t). Therefore there is an integral ideal $\mathfrak{e} \subset \mathfrak{O}_K$ such that $\mathfrak{c}\mathfrak{e} = (t)$. Observe now, that $(t)\mathfrak{A} = (t)\mathfrak{b}/\mathfrak{c} = \mathfrak{c}\mathfrak{e}\mathfrak{b}/\mathfrak{c} = \mathfrak{e}\mathfrak{b} \subset \mathfrak{O}_K$. Therefore $\mathfrak{A}$ is equivalent to $\mathfrak{b}\mathfrak{e} \subset \mathfrak{O}_K$, and we are done.

The next result(which I don't prove here) is that for any integer $x \geq 0$, the number of integral ideals $\mathfrak{a} \subset \mathfrak{O}_K$ for which $N(\mathfrak{a}) \leq x$ is finite. If this result is granted, then the number of equivalence classes of ideals is finite. Indeed, we have shown that each equivalence class of ideals can be represented by an integral ideal. This integral ideal is equivalent to another integral ideal with norm less than or equal to a given constant C_K. Using the previous(unproven result), we are done.