Chapter 1 of Knapp's "Representation Theory" is mostly review from the author's text on introductory Lie Groups. It consists of discussions on representations of compact groups and topological groups. The salient results in this chapter are Schur's Lemma on compact groups and topological groups, the Parseval-Plancharel formula for functions in $L^2(G)$. Many of the ideas here were familiar to me from the finite group case, but Knapp deserves credit for working out some character relations for compact groups–especially Schur orthogonality and the fact that characters form an orthonormal basis of class functions on the group. For Schur orthogonality, we have a compact topological group G and dx denotes Haar measure normalized so that $\int_G dx = 1$. The orthogonality relations assert that if ϕ and $\phi^{'}$ are two inequivalent irreducible unitary representations on finite dimensional vector spaces V and $V^{'}$, respectively. Then $\int_G (\phi(x)u, v), \overline{(\phi'(x)u', v')} = 0$. Also, $\int_G (\phi(x)u_1, v_1), \overline{(\phi(x)u_2, v_2)}dx = 0$. This holds for all u, v in V and all $u^{'}, v^{'}$ in V Induced representations and Frobenius reciprocity(both seminal in subsequent representation theory) are also introduced. Frobenius reciprocity says that given a closed subgroup H of G, and ϕ an irreducible representation of H on V^{ϕ} together with τ an irreducible unitary representation of G onV^{τ}, and let $\varphi = Ind_H^G\phi$ act on V^{φ}. Then $[Ind_H^G\phi : \tau] = [\tau_H : \phi]$. Knapp also introduces the Peter Weyl theorem, which gives us the density of matrix coefficients within $L^2(G)$. It concludes with the important result that any compact connected Lie Group G can be realized as a linear connected reductive Lie Group.

Chapter 2 introduces Weyl's unitary trick and explains its utility in facilitating the representation theory of various groups and algebras. There is some review of highest weight theory and the irreducible finite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$. The highest weight theorem asserts that for each integer $m > 1$, there exists up to equivalence a unique irreducible complex-linear representation π of $\mathfrak{sl}(2, \mathbb{C})$ on a space V of dimension m. In V there is a basis $v_0, ...v_{m-1)}$ such that with $n = m - 1$ we have

$(1)\pi(h)v_i = (n - 2i)v_i$

$(2)\pi(e)v_0 = 0$

$(3)\pi(f)v_i = v_{i+1}$

$(4)\pi(e)v_i = i(n - i + 1)v_{i-1}$ with $v_{i-1} = 0$.

Knapp then goes on to define Principal series and outlines the proof that these principal series representations are irreducible unitary representations of $SL(2, C)$.

The principal series are constructed as follows. They are a family of representations in $L^2(C)$ indexed by pairs (k, iv) with k an integer and v a real number. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

The representation $P^{k,iv}$ is given by $P^(k, iv)M$ acts on $f(z)$ via $|-bz+d|^{-2-iv}(-bz + d)/| - bz + d|)^{-k} f(az-c/ - bz + d)$. It is worth noting that for each pair (k, iv), $P^{k,iv}$ is an irreducible unitary representation of $SL(2, C)$ and that $P^{k,iv}$ is unitarily equivalent with $P^{-k,-iv}$.

Irreducible unitary representations of $SL(2, \mathbb{R})$ are constructed in the next section. Knapp does a decent job of highlighting the importance of $SU(1, 1)$,considering it plays a very large role later on in the text. $SU(1, 1)$ is conjugate to $SL(2, \mathbb{R})$ within $SL(2, \mathbb{C})$. One passes from representations of $SL(2, \mathbb{R})$ to representations of $SU(1, 1)$ using a change of variables and a multiplier.

Chapter 3 is the best written chapter in the book. Here the Universal Enveloping Algebra is introduced, as well as the Poincare-Birkhoff-Witt Theorem. What separates this discussion from the discussion in Knapp's introductory text on Lie Groups is the realization that the Universal Enveloping Algebra allows one to use higher derivatives. Specifically, in a Lie Algebra $\mathfrak{g}$ one can realize elements of $\mathfrak{g}$ as left invariant differential operators on the group G. The Universal enveloping algebra allows one to consider higher order differential operators also. In the third section, Knapp explains the procedure for associating to a representation of a Lie Group G on a Hilbert Space V a representation of $U(\mathfrak{g}^{\mathbb{C}})$ on the subspace of C^{∞} vectors. More precisely, if ϕ is a representation of a Lie Group G on a Hilbert space V, then we can define a linear mapping from $C^{\infty}(\phi)$ into V by

$\tau(X)(v) = lim_{t\to\infty}\phi(exptX)v - v/t.$

Each $\tau(X)$ leaves $C^{\infty}(\phi)$ stable and τ is a representation of $\mathfrak{g}$ on $C^{\infty}(\phi)$. Consequently, τ extends to a representation of $U(\mathfrak{g}^{\mathbb{C}})$

This seems to be a fairly important notion in both this text and in Varadarajan's book. The Garding subspace for Φ(a representation for the Lie Group G on a Hilbert space V) is the vector subspace of V spanned by all vectors of the form

$\Phi(f)v = \int_G f(g)\Phi(g)vdg$ for v in V and f in C^{∞}_{com}.

This result also has the important consequence that if Φ is a representation of G on a finite dimensional vector space V, then Φ is necessarily smooth as a mapping of G into $GL(V)$.

Knapp concludes the chapter with the density of the Garding subspace in a representation of the Lie group G on a Hilbert space V.

Chapter 4 is concerned with structure theory of compact groups and root-space decompositions. This was mostly review, but since I learned a bulk of this material from Knapp's other text, it is only fair that I give the man credit where it is due. Knapp drives the theory forward with examples, beginning with $sl(2,\mathbb{C}")$, and the Cartan subalgebra $\mathfrak{h}$ being the diagonal matrices. The roots $e_i - e_j$ fall out of the action of the Cartan subalgebra on the Lie algebra, and we have a reasonable understanding of the roots in type A. Knapp handles the Lie algebras of type B,C, and D as lucidly. The rest of the chapter is devoted to abstract vector spaces, dynkin diagrams, and Weyl Groups. Most of this was also familiar to me from books on reflection groups and coxeter groups. I don't think I have properly assimilated the realization of the Weyl Group analytically. Knapp finishes the chapter with Verma modules and the remarkable result that all irreducible finite dimensional representations of a semisimple Lie Algebra $\mathfrak{g}$ can be realized as submodules or quotients of the Verma module.

Chapter 5 is quite important–it introduces Cartan's decomposition as well as Iwasawa decomposition. Both of which are deeply rooted in Varadarajan's text and Knapp's. Knapp does a reasonable job of explaining the equivalence of Iwasawa decomposition with Gram-Schmidt orthogonalization for the case of $G = SL(m,\mathbb{C})$, and goes on to prove Iwasawa decomposition for linear connected semisimple groups. Iwasawa decomposition states that for a linear,connected,semisimple group G with maximal compact subgroup K and A and N analytic subgroups with Lie Algebras $\mathfrak{a}$ and $\mathfrak{n}$, A, N, and AN are simply connected closed subgroups of G and the multiplication map KAN to G is a surjective diffeomorphism.

Chapter 7 has to do with induced representations and Bruhat theory. Knapp does a decent job of pointing out why Bruhat theory is important. Specifically, given two subgroups H and$K$ of the group G, and representations p_1 and p_2 of H and K, respectively one can understand an intertwining operator between the two induced representations in terms of p_1 and p_2. This is explained using the usual machinery of double cosets. I honestly don't remember the details of the explanation, and am too lazy to open Knapp right now. The exposition devolves a little bit midway through the section. Knapp tries and fails(miserably) to motivate the Gindikin-Karpalevic formula, but all that the reader retains on a first pass through is a cumbersome integration formula. The Gindikin-Karpalevic formula starts with MAN, $MAN^{'}$ and $MAN^{''}$ parabolic subgroups with the same MA, and $\mathfrak{n}^{''} \cap \mathfrak{n} \subset \mathfrak{n}^{'} \cap \mathfrak{n}$. Define H and ρ relative to the two decompositions $G = KMAN$ and $G = KMAN^{'}$, calling them H and ρ, $H^{'}$ and $\rho^{'}$, respectively. If v is real valued and if the Haar measures are normalized, then

$$\int_{\overline{N}\cap N}(e^{-(v+\rho)H(\bar{(n)})}d\bar{(n)} = [\int_{\overline{N'}\cap N^{''}}(e^{-(v+\rho^{'})H^{'}(\overline{n^{'}}}))][\int_{\overline{N}\cap N'}(e^{-(v+\rho)H(\overline{n}}) d\overline{n}]$$

Knapp's motivation for studying these integrals is that in the induced representation space for $U(S, 1, v)$, the function obtained by extending the constant function 1 on K to G is $f(kman) = e^{-(v+\rho)log(a)}$, the left hand side of the Gindikin-Karpalevic formula is an intertwining operator of f evaluated at 1. The proposition gives a product formula for this integral, and we see that iteration of the product formula reduces the integral to a situation where it can be evaluated.

The problem with formal intertwining operators is that they're not convergent except in a narrow region where the matrix coefficients are bounded. Thus, one needs to analytically continue these operators in the variable v. Knapp introduces spherical functions in the later sections as a tool used to measure the size of matrix coefficients of various representations. Understanding these matrix coefficients is instrumental in understanding the classification of various representations. With notation as before, we define

$\phi_v^G(g) = \int_K (e^{-(v+\rho)H(g^{-1}k)}dk$ and call it a spherical function on G

Chapter 8 is probably the most important chapter of the book(so far). Knapp uses Frobenius reciprocity to show that induced representations from parabolic subgroups are admissible. There is some good writing here–specifically in showing that every K-finite vector is a C-$infinity$ vector and that the space of K-finite vectors is stable under the action of the Lie Algebra $\mathfrak{g}$. This is an interesting observation, given that the space of K-finite vectors is NOT stable under the action of the Lie Group G. Knapp then defines the Casimir element and proves that it lies in the center of the Universal Enveloping Algebra. Specifically, given a linear connected reductive group G, and real part of the trace form $C(X, Y) = Re(Tr(X, Y))$, we choose a basis $X_1, ..., X_n$ of $\mathfrak{g}$ and let $g_{ij} = C(X_i, X_j)$. Since the trace form is nondegenerate, (g_{ij}) is nonsingular. If g^{ij} denotes the inverse matrix, we set $X^j = \sum g^{ij} X_i$. Then, the Casimir element of $U(\mathfrak{g}^{\mathbb{C}})$ is defined as $\Omega = \sum g_{ij} X^i X^j$. Note that this definition is independent of basis(!). For x in G,we have $Ad(x)\Omega = \Omega$. From this, we see that the Casimir element does indeed lie in the center of the Universal Enveloping Algebra. Furthermore, using an analogue of Schur's Lemma, one may show that the center acts as a scalar on any admissible representation. This motivates the subsequent discussion of the Harish-Chandra homomorphism. Through this homomorphism, as well as infinitesimal characters(which arise as compositions of the Harish-Chandra homomorphism), we can get a grasp on the kind of scalar the center will act on an irreducible admissible by. To definie the Harish-Chandra homomorphism, we fix a positive system Δ^+ for $\Delta(\mathfrak{h}^{\mathbb{C}}, \mathfrak{g}^{\mathbb{C}})$ and let $\mathfrak{H} = U(\mathfrak{h}^{\mathbb{C}})$ and $\mathfrak{P} = \sum U(\mathfrak{g}^{\mathbb{C}})E_\alpha$ where the sum is taken over the positive roots. Let γ_{Δ^+} be the projection of $Z(\mathfrak{g}^{\mathbb{C}})$ into $\mathfrak{H} \bigoplus \mathfrak{P}$ into the $\mathfrak{H}$ factor. Now, let $\sigma_{\Delta^+}(H) = H - \delta(H)1$ where δ denotes the half sum of the positive roots. The Harish-Chandra homomorphism is obtained by composing these two mappings, and furthermore, it is an algebra isomorphism of the center of the Universal Enveloping Algebra onto those members of $\mathfrak{H}$ fixed under the action of the Weyl group.

It is worthwhile to work out the example for $\mathfrak{sl}(2,\mathbb{C})$. Here, $\Omega = 1/2h^2 + ef + fe$. With a bit of ingenuity, we can rewrite this as $\frac{1}{2}h^2 + h + 2fe$.Now $\gamma_{\Delta^+}(\Omega) = \frac{1}{2}h^2 + h$. Furthermore $\sigma_{\Delta^+}(h) = h - 1$. Since composition of these two maps yields the Harish-Chandra homomorphism, the Harish Chandra homormorphism applied to the Casimir element gives $\frac{1}{2}h^2 - \frac{1}{2}$, and we go on to see that the center of the Universal Enveloping Algebra is generated by the Casimir element.

The next two sections confused me to no end, so I omit mention of them here. The applications are numerous, beginning with the subrepresentation theorem, The subrepresentation theorem asserts that every irreducible admissible representation of a linear connected Lie group G is infinitesimally equivalent with a subrepresentation of some nonunitary principal series representation. The next application is to analytic continuation of Intertwining operators. There are other tighter classification results(asymptotic expansions near the walls as well as the asymptotic size of matrix coefficients).

Chapter 9 is devoted to the construction of discrete series. We need to understand discrete series in order to understand all irreducible admissible representations. Knapp details the $SU(1,1)$ construction. There are a few important ideas...in the compact case, the Peter-Weyl theorem tells us that each row of matrix coefficients of a representation p gives an invariant subspace of the right regular representation of $L^2(G)$ of type p. It becomes useful to understand the left regular representation of GxG on G, for then we can associate G with $GxG/diag(G)$. We want to develop a similar theory for noncompact groups G. If G^* defines the aforementioned quotient, then the center of the Universal Enveloping Algebra can be realized as an algebra of differential operators of G^*. We can seek out discrete series by looking for left KxK-finite functions on G^* that are eigenfunctions of all left GxG-invariant differential operators on G^*. This problem is attacked via duality(transferring the problem into a similar problem for K^* and G^*. Knapp uses this theory to construct discrete series and introduces the Harish-Chandra parameter as well as the Blattner parameter.

Chapter 10 introduces global characters, and many important results emerge in the introduction. The problem that one faces is that for an infinite dimensional representation π, the series defining the trace $\sum(\pi(x)v_j, v_j)$, with v_j an orthonormal basis doesn't converge. One says that a linear operator L on the Hilbert space V is of trace class is $|(B^{-1}LBv_i, v_i)|$ is finite for every orthonormal basis v_i and every bounded linear B with a bounded linear inverse. The above summation is called the trace of L. An admissible representation π of a linear connected reductive group G has global character Θ if $\pi(f)$ is of trace class for all f in C^∞_{com}. All irreducible unitary representations of a linear connected reductive group G have characters as do irreducible admissible representations. Given two infinitesimally equivalent admissible representations, Knapp demonstrates the equality of their characters. Also, global characters of infinitesimally inequivalent irreducible admissible representations are linearly independent. Knapp works out character calculations for $SL(2,R)$, deriving the character of nonunitary principal series representations of $SL(2,R)$ and shows that these are locally integrable functions. There is an important section on characters of induced representations. The setup is that G is a linear connected semisimple group, $S = MAN$ is a parabolic subgroup. If p is an irreducible unitary representation of M with specified character, the the global character of the induced representation $U(S, p, v)$ is a locally integrable function nonvanishing only on Cartan subgroups of G that are G-conjugate to Cartan subgroups of MA. Knapp goes on to explain that the restriction of any irreducible global character of G to the regular set of G is a real analytic function invariant under conjugation. Invariant eigendistributions(invariant under conjugation and where the center of Universal Enveloping Algebra acts as a scalar) is given on all of G by a locally integrable function. This is useful, because we can now show that there are only finitely many irreducible admissible representations with a given infinitesimal character. Section 9 of the chapter has to do with families of admissible representations, and attempts to answer the question of whether global characters can be generated by toying with the infinitesimal characters. This involves Zuckerman tensoring as well as Harish-Chandra modules. Zuckerman tensoring requires tensoring with finite-dimensional representations and projecting according to the effect of the center of the Universal Enveloping Algebra. Harish-Chandra modules are finitely generated admissible representations. Knapp looks at the behavior of $Z(\mathfrak{g}_{\mathbb{C}})$ on Harish-Chandra modules. The big result is that every Harish-Chandra module has a finite composition series. Therefore it has a global character that equals the sum of the global characters of its irreducible subquotients. This global character is given by a locally integrable function that is real analytic on the regular set of G. Also, any subrepresentation or quotient representation of a Harish-Chandra module is again going to be a Harish-Chandra module.

Chapter 11 is quite demanding. Knapp outlines the Plancherel formula for $SU(2)$. This states that

$\int_{SU(2)} |F(x)|^2 = \sum (n+1) \|\Phi_n(F)\|^2$ for F in $L^2(SU(2))$

This is fairly easily generalized so that it applies to any compact connected Lie group. The goal, then, is to establish an analog for noncompact semisimple groups. $SL(2,\mathbb{C})$ is the natural starting point from which one should develop this theory, and a Plancherel formula is obtained...using normalization of Haar measure and various character formulas. $SL(2,\mathbb{R})$ poses a tougher problem–here we have two nonconjugate Cartan subgroups, though the same principles apply(character formulas as well as normalization of Haar measure). The chapter closes with Hirai's patching condition which tells us what functions on G' invariant under conjugation and which serves as an eigenfunction for the center of the Universal Enveloping Algebra ensure that it is the restriction from an invariant eigendistribution on adjacent Cartan subgroups(what does it mean for two Cartan subgroups to be adjacent?)

In Chapter 12, Knapp sets about establishing that the representations constructed in the previous chapter(s) exhaust the discrete series when rank G=rank K, and that there aren't any discrete series when the rank doesn't match up with that of the maximal compact. There is a substantial theorem at the beginning which asserts that the numerator of the global character of the discrete series with specified Harish-Chandra parameter is bounded. The machinery used to show that the previous representations exhaust the discrete series consists mainly of patching conditions, and anlytically integral forms. Knapp's discussion in section 3 of the chapter is especially helpful because he gives explicit formulae for the numerator of the discrete series character. This leads to an important lemma stating that the K-character of an irreducible admissible representation exists as a distribution on K and is given on the regular set by a real analytic function. Don't really have a good command of the material surrounding the formal Dirac operator. In addition to the big theorem that a linear connected semisimple group G has discrete series representations iff rank G=rank K, there is another interesting result that states(under the same hypotheses on rank) that any given K type occurs in only finitely many discrete series, and that the trivial K type appears in no discrete series unless K is compact. Finally, there is a dicussion of tempered distributions. These are useful, because if the global character of an irreducible admissible is a tempered distribution, then the irreducible admissible is tempered(and vice versa).

Chapter 14 returns to ideas discussed much earlier in the book. Knapp wants to utilize the Langland's classification–this sets up a 1-1 correspondence between equivalence classes of irreducible admissible representations of G with triples $(S, [w], v)$, where w is an irreducible unitary representation of M, and brackets indicate its equivalence class, and S is a parabolic subgroup containing a specified parabolic subgroup of G. Thus, we need a better grasp on irreducible tempered distributions. On a pedagogical note, would it have hurt Knapp to discuss Plancherel measure for finite groups? There are some beatiful results in that setting and they make the subsequent theory so much more inviting! Knapp explains the use of intertwining operators in linking reducibility of induced representations with structure of the Plancherel measure. He proceeds with some gruesome calculations for $SL(2,\mathbb{R})$, none of which made sense to me. Eisenstein integrals are defined, and the relationship of these integrals to the matrix coefficients of the group is explained. The problem with Knapp's writing is that he makes no effort to persuade the reader of anything. Results are thrown at me and left for me to interpret. Occasionally he will dignify the exposition with a link to another branch of theory, but very seldom. After discussing asymptotics of Eisenstein integrals, Knapp lists some irreducibility results. One in particular caught my eye–if $S = MAN$ is a parabolic subgroup and p a discrete series representation of M, then $U(S, p, v)$ is irreducible for all regular imaginary v. The next section discusses the motivation behind normalizing intertwining operators as a means of attacking reducibility. The proof of Harish-Chandra's completeness theorem seems to involve many ideas I skipped over in my first pass through the material–Schwartz spaces and averaged versions of induced representations. I skipped over the intermediate sections up to section 10. There is an interesting group–$R_{u,v}$. Loosely speaking, this group R is a subgroup of a certain Weyl group and R preserves certain postive roots. This group R controls reducibility of the standard induced representations. The irreducible components of the aforementioned representations are irreducible tempered representations, and these are crucial in the Langland's classification. For this reason, the group R is relevant to our discussion. The big result bounding the size of R is in the next section–if G is a linear connected reductive group with compact center, and MAN is a parabolic subgroup, then for u in the discrete series of M and v imaginary on $\mathfrak{a}$, R sits inside the intersection of S with $W_{u,v}$. I perused the material on Schmid identities and Zuckerman tensoring, before jumping ahead to the big classification results and nondegenerate data...a basic character with nondegenerate data is irreducible if and only if its R group is trivial. Also, given a linear connected reductive group with compact center, every irreducible tempered character is basic and can be written with nondegenerate data. Also, it's R group is trivial. The converse holds as well.

Chapter 16 describes unitary representations for $SL(2,\mathbb{C})$ and $SL(2,\mathbb{R})$. Sadly $SL(2,\mathbb{C})$ has no discrete series representations because it's rank doesn't match up with that of the maximal compact subgroup. Indeed, the only irreducible unitary representations up to unitary equivalence are the trivial representation, the unitary principal series and the complementary series. For $SL(2,\mathbb{R})$, the only irreducible unitary representations up to unitary equivalence are the trivial,discrete series, irreducible members of the unitary principal series, and the complementary series.