Applications of Spectral Theory of Automorphic Forms

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Introduction

The relevance of the spectral theory of automorphic forms to number theory is powerfully illustrated by the following example.

In 1977, Haas numerically computed eigenvalues λ of the invariant Laplacian

$$\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}).$$

on $\Gamma\backslash\mathfrak{H}$, parametrized as $\lambda_w = w(w-1)$. Haas listed the w-values. Haas thought he was solving the differential equation $(\Delta - \lambda)u = 0$. Stark and Hejhal observed zeros of ζ and of an L-function on Haas' list. This suggested an approach to proving the Riemann Hypothesis, since it seemed that zeros w of ζ might give eigenvalues $\lambda = w(w-1)$ of Δ. Since Δ is a self-adjoint, nonpositive operator, these eigenvalues would necessarily be nonpositive, forcing either Re(w) $= \frac{1}{2}$ or $w \in [0, 1]$. Hejhal attempted to reproduce Haas' results with more careful computations, but the zeros failed to appear on Hejhal's list!

Hejhal realized that Haas had solved the more tolerant equation $(\Delta - \lambda)u = C \cdot \delta_\omega^{afc}$ where C is a constant, allowing a multiple of automorphic Dirac δ on the right hand side. However, since solutions u_w of $(\Delta - \lambda)u = \delta_\omega^{afc}$ are *not* genuine eigenfunctions of the Laplacian, this no longer implied nonpositivity of the eigenvalues.

For context, a solution of $(\Delta - \lambda_w)u_w = 0$ in $L^2(\Gamma\backslash\mathfrak{H})$ is either a cuspform or a constant. There is a continuous spectrum spanned by pseudo-Eisenstein series, and there are nice but not L^2 Δ-eigenfunctions, Eisenstein series. While Eisenstein series are not square-integrable, they are still eigenfunctions for Δ, with the arithmetic significance that linear combinations of values at special points give ratios of zeta functions.

The natural question was whether the Laplacian could be tweaked to overlook the intrusive distribution. That is, one would want a variant $\Delta_?$ for which $(\Delta_? - \lambda_w)u_w = 0$ whenever $(\Delta-\lambda_w)u_w = \delta_\omega^{afc}$. Because of Colin de Verdiere's argument for meromorphic continuation of Eisenstein series, where a different distribution was overlooked, it was anticipated that $\Delta_? = \Delta^{\mathrm{Fr}}$ would be a fruitful choice for a suitably chosen Friedrichs extension. Δ^{Fr} is self-adjoint and therefore symmetric. This gave glimpses of a potential RH proof again!

Friedrichs extensions have the desired properties. Classically, Friedrichs extensions make self-adjoint operators out of symmetric operators. The (modern) bonus is that they can be constructed so as to overlook distributions such as Dirac delta. Friedrichs extensions played a big part in another story, namely Colin de Verdiere's meromorphic continuation of Eisenstein series, though in that story, they overlooked a different distribution.

There, the spaces of interest were the orthogonal complements $L^2(\Gamma\backslash\mathfrak{H})_a$ to the space spanned by pseudo-Eisenstein series whose test function is supported on $[a, \infty)$. Δ_a was Δ with domain $C_c^\infty(\Gamma\backslash\mathfrak{H})$ and constant term vanishing above height $y = a$. Δ_a^{Fr} was the Friedrichs extension of Δ_a to a self-adjoint unbounded operator on $L^2(\Gamma\backslash\mathfrak{H})_a$.
This Friedrichs extension overlooks the distribution on $\Gamma\backslash\mathfrak{H}$ given by

$$T_a(f) \quad = \quad (c_p f)(ia).$$

To explain the context of automorphic spectral expansions of automorphic distributions, it is worth remarking that in classical Fourier analysis, $\delta = \sum_n 1 \cdot e^{2\pi i n x}$ converges neither pointwise nor in $L^2(\mathbb{R}/\mathbb{Z})$. However, it does converge meaningfully in a negatively indexed Sobolev space $H^s(\mathbb{R}/\mathbb{Z})$ for $s < -\frac{1}{2}$. Similarly representation theory shows that δ^{afc} lies in a suitable global automorphic Sobolev space. The numerology in Sobolev spaces indicates why certain things aren't possible and why previous attempts failed.

The Friedrichs construction automatically produces all eigenfunctions inside a $+1$-index Sobolev space. The Dirac δ on a two dimensional manifold is in Sobolov space with index $-1-\epsilon$ for all ϵ bigger than (but not equal to) 0, so by elliptic regularity a fundamental solution is in the $+1-\epsilon$ index Sobolev space, and definitely *not* in the $+1$ index Sobolev space. This implies that the fundamental solution couldn't possibly be an eigenfunction for *any* Friedrichs extension of a restriction of Δ described by boundary conditions. This was *not* understood quite so clearly thirty years ago.

This gives us a compelling reason to study the spectral theory of automorphic forms, as they encode simple yet elegant number theoretic phenomenon. As rich as the SL_2 configuration is, it isn't indicative of the complexity of higher rank groups. Indeed, for SL_2, the residual spectrum of the Laplacian consists only of constants. For SL_4, there is a marked difference, in that Speh forms also enter into the residual spectrum. This provides an incentive for setting up a finer harmonic analysis on higher rank groups; in particular, one that does not use gritty details.

1 Spectral Theory of Automorphic Forms

Let $G = SL_2(\mathbb{R})$. G acts transitively on the upper-half plane $\mathfrak{H}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

The isotropy group of $i \in \mathfrak{H}$ is $K = SO_2(\mathbb{R})$, so $G/K \cong \mathfrak{H}$ as G-spaces. Let $\mathfrak{g}$ denote the Lie algebra of G, and $U\mathfrak{g}$ the universal enveloping algebra. Construct the simplest non-trivial G-invariant element, the **Casimir** element in $U\mathfrak{g}$. This is the image of $1_{\mathfrak{g}}$ under the chain of G-equivariant maps

$$\mathrm{End}_{\mathbb{R}}(\mathfrak{g}) \to \mathfrak{g} \otimes \mathfrak{g}^* \to \mathfrak{g} \otimes \mathfrak{g} \to A\mathfrak{g} \to U\mathfrak{g}.$$

The first map is the natural isomorphism, the second is an isomorphism via the trace pairing on the second factor. The third map is inclusion, while the fourth is through imbedding to the quotient. It is computationally necessary to work out the Casimir element in coordinates. For any basis $x_1, ..., x_n$ of a semisimple Lie algebra $\mathfrak{g}$ let $x_1^*, ..., x_n^*$ be the corresponding dual basis relative to the trace pairing $\langle X, Y\rangle = \mathrm{Tr}(XY)$. The Casimir element is

$$\sum_i x_i x_i^* \in Z\mathfrak{g}.$$

where $Z\mathfrak{g}$ is the center of the universal enveloping algebra, and the Casimir element is $\mathrm{Ad}(G)$-invariant. The Lie algebra $\mathfrak{g}$ naturally maps to an algebra of differential operators on the space $C_c^\infty(G)$ as follows. For $x \in \mathfrak{g}$ and $F \in C_c^\infty(G)$,

$$(x \cdot F)(g) = \frac{d}{dt}\bigg|_{t=0} F(g\, e^{tx}).$$

is a left G-invariant differential operator

$$F(h \cdot (g\, e^{tx})) = F((h \cdot g) \cdot e^{tx}) \quad (\text{for } g, h \in G,\ x \in \mathfrak{g}).$$

However, there is no notion of composition in the Lie algebra that maps to composition of differential operators. There is a notion of associative composition in the universal enveloping algebra. It is computationally useful to know how the Casimir operator looks as a differential operator on G/K.

For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$, a standard choice of basis elements is

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

with commutation relations $[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H$. Relative to the trace pairing,

$$\langle H, H\rangle = 2 \quad \langle H, X\rangle = 0 \quad \langle H, Y\rangle = 0 \quad \langle X, Y\rangle = 1.$$

Therefore, Casimir is $\Omega = \frac{1}{2}H^2 + XY + YX$. This may be rewritten as

$$\Omega \quad = \quad \frac{1}{2}H^2 + XY - (-YX) \quad = \quad \frac{1}{2}H^2 + 2XY - (XY - YX) \quad = \quad \frac{1}{2}H^2 + 2XY - H.$$

To make a G-invariant differential operator on $\mathfrak{H}$, use the G-space isomorphism $G/K \cong \mathfrak{H}$. Let $q : G \to G/K$ be the quotient map

$$q(g) = gK \to g(i).$$

A function f on $\mathfrak{H}$ naturally yields the right K-invariant function $f \circ q$

$$(f \circ q)(g(i)) \quad = \quad f(g(i)) \quad \text{(for } g \in G).$$

The computation of Ω on $f \circ q$ can be simplified by using the right K-invariance of $f \circ q$ which means that $f \circ q$ is annihilated by

$$\mathfrak{so}_2(\mathbb{R}) \quad = \quad \text{skew-symmetric 2-by-2 real matrices} \quad = \quad \{\begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix} : t \in \mathbb{R}\}.$$

So in terms of our Lie algebra basis, $X - Y$ annihilates $f \circ q$. A point $z = x + iy \in \mathfrak{H}$ is the image

$$x + iy \quad = \quad (n_x \cdot m_y)(i) \ \text{ where } \ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \ \ m_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}.$$

These are convenient group elements because they match exponentiated Lie algebra elements:

$$e^{tX} \quad = \quad n_t \qquad e^{tH} \quad = \quad m_{e^{2t}}.$$

Since $X - Y$ acts trivially on right K-invariant functions on G, the action of Y is the same as the action of X on right K-invariant functions. Observe that

$$(X \cdot F)(n_x m_y) \quad = \quad \frac{d}{dt}\Big|_{t=0} F(n_x m_y n_t) \quad = \quad \frac{d}{dt}\Big|_{t=0} F(n_x n_{yt} m_y) \quad = \quad \frac{d}{dt}\Big|_{t=0} F(n_{x+yt} m_y) \quad = \quad y\frac{\partial}{\partial x}F(n_x m_y).$$

Thus, the term $2X^2$ gives

$$2X^2 \to 2y^2(\frac{\partial}{\partial x})^2.$$

Similarly, the action of H is

$$(H \cdot F)(n_x m_y) \quad = \quad 2y\frac{\partial}{\partial y}F(n_x m_y).$$

Then

$$\frac{H^2}{2} - H \quad = \quad 2y^2(\frac{\partial}{\partial y})^2.$$

Altogether,

$$\Omega \quad = \quad y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}).$$

A **cuspform** is a $\mathbb{C}$-valued function on $\Gamma\backslash\mathfrak{H}$ which is an eigenfunction for the $SL_2(\mathbb{R})$-invariant $\Delta = y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$, and has constant term (explained later) equal to 0. Around 1956, Roelke and Selberg ([Roelke 1956], [Selberg 1956]) proved that $L^2(\Gamma\backslash\mathfrak{H})$ decomposes discretely into cuspforms, constants, and continuous spectrum.

Plancherel/Spectral Expansion Let $X = \Gamma\backslash\mathfrak{H}$, $\Gamma = SL_2(\mathbb{Z})$, $G = SL_2(\mathbb{R})$, and $K = SO(2)$. Functions $f \in L^2(\Gamma\backslash\mathfrak{H})$ decompose in an L^2 sense

$$f \quad = \quad \sum_F \langle f, F\rangle \cdot F + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \langle f, E_s\rangle \cdot E_s.$$

where F runs over an orthonormal basis of cuspforms, as defined below, making up the bulk of the discrete spectrum. The Eisenstein series E_s are non L^2-eigenfunctions. They will be described in detail below. Plancherel holds here

$$\|f\|_2^2 \quad = \quad \sum_F |\langle f, F\rangle|^2 + \frac{|\langle f, 1\rangle|^2}{\langle 1, 1\rangle} + \frac{1}{2\pi}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} |\langle f, E_s\rangle|^2.$$

For example [Garrett 2010] or [Iwaniec] prove the *standard estimates*

$$\sum_{s_F \le T} |F(z_0)|^2 + \frac{1}{4\pi}\int_{-T}^{T} |E_s(z_0)|^2 dt \quad \ll \quad T^2$$

Using integration/summation by parts, this can be interpreted as asserting that the automorphic Dirac δ at z_0 lies in the negatively-indexed global automorphic Levi-Sobolev space $H^{-1-\epsilon}$ for all $\epsilon > 0$, where the s$^{\text{th}}$ Levi-Sobolev space is

$$H^s(\Gamma\backslash\mathfrak{H}) \quad = \quad \{\sum_F a_F \cdot F + \frac{a_1 \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} a_s \cdot E_s \, ds \quad : \quad \sum_F |a_F|^2 + \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} |a_s|^2 \, ds < \infty\}.$$

The spectral expansion of automorphic δ^{afc} at base point z_0 is

$$\delta^{\text{afc}} \quad = \quad \sum_F \overline{F}(z_0) \cdot F + \frac{1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} E_{1-s}(z_0) \cdot E_s \; .$$

This converges in the $(-1-\epsilon)^{\text{th}}$ global automorphic Sobolev space for every $\epsilon > 0$. Since we do not have, and do not expect to have, pointwise estimates on either cuspforms or Eisenstein series, these considerations are rather disconnected from local Sobolev estimates.

Spectral expansions allow us to solve the natural differential equation

$$(\Delta - \lambda_w)u_w = \delta_{z_0}^{\text{afc}}.$$

by writing out the spectral expansion of u, applying $(\Delta - \lambda_w)$ to it and equating the result to the spectral expansion of $\delta_{z_0}^{\text{afc}}$ above. This gives

$$u \quad = \quad \sum_F \frac{\overline{F}(x_0) \cdot F}{\lambda_w - \lambda_F} + \frac{1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \frac{E_{1-s}(z_0) \cdot E_s}{w(w-1) - s(s-1)} \, ds.$$

This converges in the $-\epsilon^{\text{th}}$ Sobolev norm. Nevertheless, as a function-valued function with values in that Sobolev space, it has a meromorphic continuation with poles at s giving eigenvalues of cuspforms. The continuous spectrum part has poles to the left of the critical line, corresponding to zeros of $\zeta(2w)$.

Spherical Analysis on $SL_2(\mathbb{C})$

Recall the *Poisson summation formula*: let f be a Schwartz function on $\mathbb{R}$; then

$$\sum_{n\in\mathbb{Z}} f(n) \quad = \quad \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

This is proven by defining a smooth function F on $\mathbb{R}/\mathbb{Z}$ by

$$F(x) \quad = \quad \sum_{n\in\mathbb{Z}} f(x+n).$$

and equating $F(0)$ with the Fourier series of F evaluated at zero.

Poisson summation is the prototype for *trace formulas*. More specifically, let G be a unimodular group, and Γ a discrete subgroup. Let G act on $L^2(\Gamma\backslash G)$ by right translation π, and let π_1 be the representation of the algebra of continuous compactly-supported functions on G on the Hilbert space $L^2(\Gamma\backslash G)$ by

$$\pi_1(\varphi)(f) \quad = \quad \int_G \pi(h)(f)\varphi(h)\, dh.$$

Suppose that $\Gamma\backslash G$ is compact. Then the map $f \to \pi_1(\varphi)(f)$ is given by integration against a square-integrable kernel $K_\varphi(g,h)$, so that π_1 is a compact operator.

Intuitively,

$$\text{trace } \pi_1(\varphi) \quad = \quad \int_{\Gamma\backslash G} K(h,h)\, dh \quad = \quad \sum_{\{\alpha\}} \int_{\Gamma_\alpha\backslash G} \varphi(h^{-1}\alpha h)\, d\gamma.$$

where $\{\alpha\}$ denotes the conjugacy class of α and Γ_α is the centralizer of α in Γ. Letting G_α be the centralizer of α in G, this becomes

$$\text{trace } \pi_1(\varphi) \quad = \quad \sum_{\{\alpha\}} \int_{G_\alpha\backslash G} \varphi(h^{-1}\alpha h)\, d\gamma \cdot \text{vol}(\Gamma_\alpha\backslash G).$$

Because π_1 is a compact operator, $L^2(\Gamma\backslash G)$ decomposes *discretely* as

$$L^2(\Gamma\backslash G) \quad = \quad \tilde{\bigoplus} \mu_\beta V_\beta.$$

where β runs over irreducible unitary Hilbert space representations of G, and the μ_β's are integers. Therefore,

$$\sum_\beta \mu_\beta \text{ trace}\beta_1(\varphi) \quad = \quad \text{trace } \pi_1(\varphi).$$

which we know the equal to

$$\sum_{\{\alpha\}} \int_{G_\alpha \backslash G} \varphi(h^{-1}\alpha h)\, d\gamma \cdot \mathrm{vol}(\Gamma_\alpha \backslash G).$$

This is the trace formula in the compact quotient case. Note that for $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$, the trace formula gives us the Poisson summation formula. This is why the trace formula is referred to as a nonabelian Poisson summation formula.

2 Pseudo-Eisenstein series and the continuous spectrum

We review [Garrett 2011f] throughout this section. Let N be the subgroup of G of upper triangular unipotent matrices, A^+ the subgroup of diagonal matrices with positive diagonal entries, and P the parabolic subgroup of all upper-triangular matrices. The **constant term** along P of a function f on $\Gamma\backslash G$ is

$$c_P f(g) = \int_{(N\cap\Gamma)\backslash N} f(ng)\, dn.$$

Note that $c_P f$ is left N-invariant. A function f on $\Gamma\backslash G$ is a **cuspform** when $c_P(f) = 0$, treating $c_P f$ as a distribution. That is, a function f is a cuspform iff

$$\int_{N\backslash G} c_P f(g)\, \psi(g)\, dg \quad = \quad 0.$$

for all ψ in $C_c^\infty((P\cap\Gamma)N\backslash G)$. That is, the cuspform condition is that the constant term vanishes as a *distribution* on $(P\cap\Gamma)N\backslash G$

2.1 Pseudo-Eisenstein series We follow [Garrett 2011f]. Pairings

$$\langle f, F\rangle_{H\backslash G} = \int_{H\backslash G} f \cdot \overline{F} dg.$$

are hermitian (as opposed to bilinear). Given ψ in $C_c^\infty((P\cap\Gamma)N\backslash G)$ the pseudo-Eisenstein series Ψ_ψ attached to ψ, is characterized by the adjunction

$$\langle c_P f, \psi\rangle_{(P\cap\Gamma)N\backslash G} \quad = \quad \langle f, \Psi_\psi\rangle_{\Gamma\backslash G}.$$

Indeed

$$\langle c_P f, \psi\rangle_{(P\cap\Gamma)N\backslash G} = \int_{(P\cap\Gamma)N\backslash G} c_P f(g)\psi(g) dg \quad = \quad \int_{P\cap\Gamma\backslash G} f(g)\psi(g) dg \quad = \quad \int_{\Gamma\backslash G} f(g) \left(\sum_{P\cap\Gamma\backslash\Gamma} \psi(\gamma g) \right) dg.$$

Therefore,

$$\Psi_\psi(g) \quad = \quad \sum_{P\cap\Gamma\backslash\Gamma} \psi(\gamma g).$$

The sum describing the pseudo-Eisenstein series is **locally finite**: the sum has only finitely many nonzero summands for g in a fixed compact. In particular $\Psi_\psi \in C_c^\infty(\Gamma\backslash G)$.

Corollary: The square integrable cuspforms are the orthogonal complement of the closed space spanned by the pseudo-Eisenstein series in $L^2(\Gamma\backslash G)$.

We now decompose pseudo-Eisenstein series using Fourier-Mellin transforms. The **Fourier-Mellin transform** of $F \in C_c^\infty(0, +\infty)$ is

$$\mathbf{M}F(s) \quad = \quad \int_0^\infty F(r) r^{-s} \, \frac{dr}{r} \quad (\text{for } s \in \mathbb{C}).$$

Remark: It is important that for $f \in C_c^\infty(\mathbb{R})$ the Fourier transform extends to an entire function of rapid decay on vertical lines (by the Paley-Weiner theorem). The same is true for the Mellin transform since it is the Fourier transform in different coordinates.

For any real σ, Mellin inversion is

$$F(y) \quad = \quad \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathbf{M}F(s) y^s \, ds.$$

For $\varphi \in C_c^\infty(0, \infty)$, the Mellin inversion formula gives

$$\varphi(y) \quad = \quad \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathbf{M}\varphi(s) y^s \, ds.$$

This is

$$\varphi(g) \quad = \quad \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathbf{M}\varphi(s) \chi_s(a(g)) \, ds.$$

Thus, the pseudo-Eisenstein series is expressible as

$$\Psi_\varphi(g) \quad = \quad \sum_{\gamma \in (\Gamma \cap N)\backslash\Gamma} \varphi(\gamma g) \quad = \quad \frac{1}{2\pi i} \sum_{\gamma \in (\Gamma \cap N)\backslash\Gamma} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathbf{M}\varphi(s) \cdot \chi_s(a(\gamma g)) \, ds.$$

Taking $\sigma = 0$ would be natural, but, with $\sigma = 0$, the double integral (sum and integral) is not absolutely convergent, and the two integrals can not be interchanged. The best line along which to integrate is $\sigma = \frac{1}{2}$, but this is not in the region of convergence. For $\sigma > 1$, elementary estimates show that the double integral is absolutely convergent, and using Fubini, the two integrals can be interchanged:

$$\Psi_\varphi(g) \quad = \quad \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \mathbf{M}\varphi(s) \sum_{\gamma \in (\Gamma \cap N)\backslash\Gamma} \chi_s(a(\gamma g)) \, ds \quad \text{for } \sigma > 1.$$

The inner sum defines the familiar spherical Eisenstein series

$$E_s(g) \quad = \quad \sum_{\gamma \in (\Gamma \cap P)\backslash\Gamma} \chi_s(a(\gamma g)) \quad = \quad \sum_{\gamma \in (\Gamma \cap P)\backslash\Gamma} \mathrm{Im}(\gamma z)^s.$$

Therefore, the pseudo-Eisenstein series spectrally decompose as

$$\Psi_\psi \;=\; \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathbf{M}\psi(s)\cdot E_s(g)\,ds \quad (\sigma > 1).$$

this discussion leads to the formula defining the Eisenstein series in the region of convergence, namely

$$E_s(g) \;=\; \sum_{\gamma\in(\Gamma\cap P)\backslash\Gamma} \mathrm{Im}(\gamma z)^s.$$

The Eisenstein series converges only for $\mathrm{Re}(s) > 1$ and it is essential to establish meromorphic continuation, since the spectral decomposition and Plancherel require it. This will be discussed later.

3 Spectral Decomposition of Pseudo-Eisenstein series

Our goal is to rewrite the spectral decomposition to refer only to Ψ_ψ and not ψ. We review some standard adjunction relations of Eisenstein series from [Garrett 2011f]. The Eisenstein series E_s on $\Gamma\backslash\mathfrak{H}$ fits into the adjunction

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \langle y^{1-s}, c_P f\rangle_{(P\cap\Gamma)N\backslash\mathfrak{H}}.$$

This realizes integrals against Eisenstein series as Mellin transforms of constant terms:

$$\langle E_s, f\rangle_{\Gamma\backslash\mathfrak{H}} \;=\; \mathbf{M}(c_P f)(1-s).$$

With the usual G-invariant Laplacian Δ on G/K, from

$$\Delta y^s \;=\; s(s-1)\cdot y^s.$$

in the region of convergence

$$\Delta E_s \;=\; s(s-1)\cdot E_s.$$

Since Δ commutes with the map $f \to c_P f$, we see that $c_P E_s$ is a function $u(y)$ of y satisfying the Eulerian equation

$$y^2\frac{\partial^2}{\partial y^2}u(y) \;=\; s(s-1)\cdot u(y).$$

For $s \neq \frac{1}{2}$ this has the two linearly independent solutions y^s and y^{1-s}, so for some meromorphic functions a_s and c_s,

$$c_P E_s \;=\; a_s y^s + c_s y^{1-s}.$$

Proposition: The constant term of the spherical Eisenstein series is

$$c_P E_s \;=\; y^s + c_s y^{1-s} \quad \text{with } \; c_s \;=\; \frac{\xi(2s-1)}{\xi(2s)}.$$

To see this, define a function φ_v on $G_v = GL_2(\mathbb{Q}_v)$ by

$$\varphi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot k) \;=\; \left|\frac{a}{d}\right|_v^{2s}.$$

where in all cases $a, d \in \mathbb{Q}_v^\times$, $b \in \mathbb{Q}$, and k is in the standard maximal compact of $GL_2(\mathbb{Q}_v)$. Let

$$\varphi = \otimes \varphi_v.$$

Let P be upper triangular matrices in G. Given the Eisenstein series

$$E_s(g_\infty) = \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma \cdot g_\infty).$$

we compute

$$c_P E_s(g) = \int_{N_\mathbb{Q} \backslash N_\mathbb{A}} E_s(ng)\, dn.$$

Parametrizing $P_\mathbb{Q} \backslash G_\mathbb{Q}$ via the Bruhat decomposition,

$$\int_{N_\mathbb{Q} \backslash N_\mathbb{A}} E_s(ng)\, dn = \int_{N_\mathbb{Q} \backslash N_\mathbb{A}} \sum_{\gamma \in P_\mathbb{Q} \backslash G_\mathbb{Q}} \varphi(\gamma n g)\, dn = \sum_{P_\mathbb{Q} \backslash G_\mathbb{Q} / N_\mathbb{Q}} \int_{N_\mathbb{Q} \backslash N_\mathbb{A}} \sum_{\gamma \in P_\mathbb{Q} \backslash P_\mathbb{Q} w N_\mathbb{Q}} \varphi(\gamma n g)\, dn.$$

Using the Bruhat decomposition, $P_\mathbb{Q} \backslash G_\mathbb{Q} / N_\mathbb{Q}$ has exactly two representatives, 1 and w. Therefore the constant term simplifies to

$$\int_{N_\mathbb{Q} \backslash N_\mathbb{A}} \varphi(ng)\, dn + \int_{N_\mathbb{A}} \varphi(wng)\, dn.$$

The first summand simplifies to

$$\int_{N_\mathbb{Q} \backslash N_\mathbb{A}} \varphi(ng)\, dn = \varphi(g) \cdot \mathrm{vol}(N_\mathbb{Q} \backslash N_\mathbb{A}) \;\;=\;\; \varphi(g) \cdot 1.$$

The second summand can be written as a product over primes

$$\int_{N_\mathbb{A}} \varphi(wng)\, dn = \prod_{v \le \infty} \int_{N_v} \varphi_v(wng)\, dn.$$

For $g \in G_\infty$ so that $g_v = 1$, the finite-prime local factors in the Euler product for the big Bruhat cell are evaluated. Note

$$\varphi_v(w \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}) \;\;=\;\; \begin{cases} 1 & \text{for } |t|_v \le 1 \\ |t|_v^{-2s} & \text{for } |t|_v > 1 \end{cases}.$$

With the v-adic factor corresponding to the prime p, the v-adic local factor is

$$\int_{|t|_v \le 1} 1\, dt + \int_{|t|_v > 1} |t|_v^{-2s}\, dt = 1 + \sum_{l=1}^{\infty} |p^{-l}|_v^{-2s} \cdot \int_{p^{-l}\mathbb{Z}_p} 1\, dt = \frac{1 - p^{-2s}}{1 - p^{1-2s}} = \frac{\zeta_v(2s-1)}{\zeta_v(2s)}.$$

where $\zeta_v(s)$ is the v^{th} Euler factor of the zeta function. Thus, the finite prime part of the big-cell summand is $\frac{\zeta(2s-1)}{\zeta(2s)}$.

Next, we compute the archimedean factor of the big-cell summand of the constant term and get

$$\int_{\mathbb{R}} \left| \frac{y}{(x+t)^2+y^2} \right|_\infty^s dt \quad = \quad y^{1-s} \cdot \frac{\zeta_\infty(2s-1)}{\zeta_\infty(2s)}.$$

Thus, with $\xi(s)$ the completed zeta function $\xi(s) = \zeta_\infty(s) \cdot \zeta_s$, the constant term of the Eisenstein series is

$$c_P E_s(x+iy) = y^s + \frac{\xi(2s-1)}{\xi(2s)} \cdot y^{1-s}.$$

Returning to spectral theory,

$$\Psi_\psi = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathbf{M}\psi(s) \cdot E_s(g)\, ds.$$

The meromorphic continuation of Eisenstein series and some soft estimates on Eisenstein series allow the line of integration to be moved to the left to $\sigma = \frac{1}{2}$, and rewrite

$$\Psi_\psi = \frac{1}{2\pi i} \int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \mathbf{M}\psi(s) \cdot E_s(g) ds + \sum_{s_0} \mathrm{res}_{s=s_0} E_s \cdot \mathbf{M}\psi(s).$$

From the *theory of the constant term* [Moeglin-Waldspurger 1995], a moderate-growth eigenfunction for Casimir, with (standard) constant term subtracted, is of rapid decay in Siegel sets. Observe

$$c_P(E_{1-s} - \frac{E_s}{c_s}) = (y^{1-s} + c_{1-s} y^s) - (\frac{y^s + c_s y^{1-s}}{c_s}) = (c_{1-s} - \frac{1}{c_s}) y^s.$$

For $\mathrm{Re}(s) > 0$ and off the real line, the Casimir eigenvalue $s(s-1)$ is not real, yet y^s is square-integrable on Siegel sets for $0 < \mathrm{Re}(s) < \frac{1}{2}$. That is, the difference $E_{1-s} - \frac{1}{c_s} \cdot E_s$ is in $L^2(\Gamma\backslash\mathfrak{H})$. Since the Casimir operator is symmetric, any eigenvalue must be real. Therefore $E_{1-s} - \frac{1}{c_s} \cdot E_s$ is identically zero, which gives the functional equation and relation

$$E_{1-s} \quad = \quad \frac{E_s}{c_s} \qquad c_s \cdot c_{1-s} = 1.$$

Combining the adjunction property of the pseudo-Eisenstein series with the constant term of the spherical Eisenstein series: $c_P E_s = y^s + c_s y^{1-s}$,

$$\mathbf{M}(c_P \Psi_\psi)(1-s) = \mathbf{M}\psi(1-s) + c_s \mathbf{M}\psi(s).$$

Combining this with the previous expression for the pseudo-Eisenstein series gives

$$\Psi_\psi - (\text{residual part}) = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \mathbf{M} c_P \Psi_\psi(s) \cdot E_s ds = \frac{1}{2\pi i} \int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle E_{1-s}, \Psi_\psi \rangle_{\Gamma\backslash\mathfrak{H}} \cdot E_s\, ds.$$

3.2 Plancherel for $PSL(2,\mathbb{Z})$

We have a decomposition of a pseudo-Eisenstein as an integral of Eisenstein series E_s on Re(s)= $\frac{1}{2}$, plus residues

$$\Psi_\varphi - \text{residual part} = \frac{1}{2\pi i}\int_{\frac{1}{2}+i0}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_{1-s}\rangle \cdot E_s \, ds.$$

Let $f_1 \in C_c^\infty(\Gamma\backslash G), \varphi \in C_c^\infty(N\backslash G)$, and assume that Ψ_φ is orthogonal to residues of Eisenstein series, which are constants. Using the spectral decomposition of the pseudo-Eisenstein series in terms of the Eisenstein series, we obtain

$$\langle \Psi_\varphi, f\rangle = \frac{1}{4\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} \langle \Psi_\varphi, E_{1-s}\rangle \cdot \langle E_s, f\rangle \, ds.$$

Therefore the map $f \to (s \to \langle f, E_s\rangle)$ is an inner-product-preserving map from the Hilbert-space span of the pseudo-Eisenstein series to $L^2(\frac{1}{2}+i\mathbb{R})$.

The map $\Psi_\varphi \to \langle \Psi_\varphi, E_{1-s}\rangle$ produces functions $u(t) = \langle \Psi_\varphi, E_{1-s}\rangle$ satisfying

$$u(-t) \quad = \quad \langle \Psi_\varphi, E_s\rangle \quad = \quad \langle \Psi_\varphi, c_s E_{1-s}\rangle = c_s \cdot u(t).$$

It can be shown that any $u \in L^2(\frac{1}{2}+i\mathbb{R})$ satisfying $u(-t) = c_s u(t)$ is in the image. The functions $s \to \langle \Psi_\varphi, E_s\rangle$ are dense in the space of $L^2(\frac{1}{2}+i\mathbb{R})$ functions satisfying $u(-t) = c_s \cdot u(t)$. Therefore, we have an isometry

$$\{\text{cuspforms}\}^\perp \cap L^2(\Gamma\backslash G)^K \cong \{u \in L^2(\Gamma\backslash G/K) : u(-t) = c_s \cdot u(t)\}.$$

4 Friedrichs extensions

4.1 Construction

We discuss Friedrichs extensions, following [Garrett 2011c], [Friedrichs 1935a] and [Friedrichs 1935b].

Let T be a densely defined, symmetric, strictly positive operator on a Hilbert space V, with domain D. Assume further, that T is semi-bounded from below in the sense that

$$\|u\|^2 \quad \le \quad \langle u, Tu\rangle \ \text{ for all } u \in D.$$

The characterization and construction of the Friedrichs extension of T depend on D.

Let $\langle x,y\rangle_1 = \langle Tx, y\rangle$ on D. Define V_1 be the completion of D with respect to the new inner product. The operator T remains symmetric for $\langle,\rangle_1$. That is, $\langle Tx,y\rangle_1 = \langle x, Ty\rangle_1$ for $x,y \in V_1$. By Riesz-Fisher, for $y \in V$, the continuous linear functional $f(x) = \langle x,y\rangle$ can be written $f(x) = \langle x, y'\rangle_1$ for a unique $y' \in V$. The map $y \to T_{\mathrm{Fr}}^{-1} y$ is injective, so set $T_{\mathrm{Fr}}^{-1} y = y'$. That is, the inverse T_{Fr}^{-1} of the Friedrichs extension T_{Fr} of T is an everywhere-defined map $T_{\mathrm{Fr}}^{-1} y : V \to V_1$, continuous for the $\langle , \rangle_1$ topology on V_1, characterized by

$$\langle Tx, T_{\mathrm{Fr}}^{-1} y \rangle = \langle x, y \rangle.$$

Proof: Since $\langle , \rangle_1 \geq \langle , \rangle$, the completion V_1 continuously imbeds in V, extending the inclusion $D \subset V$.

For $y \in V$ and $x \in V_1$, the functional $\lambda_y x = \langle x, y \rangle$ has bound $|\lambda_y x| \leq |x| \cdot |y| \leq |x|_1 \cdot |y|_1$, so by Riesz-Fischer, there is unique $By \in V_1$ such that $\lambda_y x = \langle x, By \rangle_1$. The bound gives continuity of B. The map $y \to By$ is linear.

The map B is injective: for $By = 0$ for $y \in V$, for all $x \in V_1$

$$0 = \langle x, 0 \rangle_1 = \langle x, By \rangle_1 = \langle x, y \rangle.$$

Density of V_1 in V gives $y = 0$. Thus, B has a possibly unbounded, symmetric inverse $S = B^{-1}$, which surjects to V from its domain. Further, the domain of S is $\langle , \rangle_1$-dense in V_1 since $\langle x, By \rangle_1 = \langle x, y \rangle$ for $x \in V_1$ and $y \in V$.

Next, B is symmetric:

$$\langle Bx, y \rangle \quad = \quad \lambda_y Bx \quad = \quad \overline{\langle By, Bx \rangle_1} \quad = \quad \overline{\lambda_x By} \quad = \quad \langle x, By \rangle \text{ (for } x, y \in V).$$

Thus, being bounded, B is self-adjoint.

Next, the inverse $S = B^{-1}$ is self-adjoint. Let $\sigma : V \bigoplus V \to V \bigoplus V$ be defined by $\sigma(x \oplus y) = y \oplus x$. Certainly graph $S = \sigma$ (graph$B$). Let $U(x \oplus y) = -y \oplus x$. For any densely defined operator Φ, the graphs of Φ and its adjoint are related by

$$\text{graph } \Phi^* \quad = \quad (U \text{ graph } \Phi)^\perp.$$

U and σ have the commutation relation $U \circ \sigma = -\sigma \circ U$, so

$$\text{graph } S^* \quad = \quad (U \text{ graph } S)^\perp = (U \ \sigma \text{graph } B)^\perp = (-\sigma U \text{ graph } B)^\perp.$$

Since $(\sigma X)^\perp = \sigma(X^\perp)$ in general, and since $B^* = B$,

$$(-\sigma U \text{ graph } B)^\perp \quad = \quad -\sigma(\text{graph } B^*) \quad = \quad -\sigma(\text{graph } B) \quad = \quad -\text{graph } S \quad = \quad \text{graph } S.$$

That is, graph $S^* =$ graph S, giving the self-adjointness of S.

Next, we show that $\langle x, Sx \rangle \geq \langle x, x \rangle$ for x in the domain of S. Every x in the domain of S is of the form $x = By$ for some $y \in V$, so

$$\langle x, Sx \rangle \quad = \quad \langle By, SBy \rangle \quad = \quad \langle By, y \rangle \quad = \quad \lambda_y By \quad = \quad \langle By, By \rangle_1 \geq \langle By, By \rangle \geq \langle x, x \rangle.$$

To see that S extends T, first show that the domain of S contains D, the domain of T. From $\langle x, y \rangle_1 = \langle x, Ty \rangle = \langle x, BTy \rangle_1$ for $x \in V_1$ and $y \in D$, necessarily $\langle x, y - BTy \rangle = 0$.

Thus, $BTy = y$ for $y \in D$, and, in particular, y is in the range of B, which is the domain of S. Then it is legitimate to compute

$$Sx \;=\; S(BT)x \;=\; (SB)Tx = Tx \;\text{ for } x \in D.$$

This completes the proof.

4.2 Friedrichs extensions of restrictions

In applications, suitably designed Friedrichs extensions of restrictions of natural differential operators effectively ignore certain distributions. This is traditionally applied to boundary-value problems. For example, the Friedrichs extension Δ_{Fr} of the Laplacian Δ, with domain test functions $C_c^\infty(U)$ on a bounded open set U in $\mathbb{R}^n$, with smooth boundary, is related to the boundary-value problem $\Delta u \;=\; f$ with u vanishing on the boundary of U. We claim that for u smooth on the interior of U, vanishing on the boundary, and vanishing outside U, $\Delta_{\mathrm{Fr}} u = f$ must mean $\Delta u = f + \theta$ for a distribution θ supported on the boundary of U and in the Sobolev space $H^{-1}(\mathbb{R}^n)$.

Complex conjugation maps As in the example of complex conjugation of almost-everywhere pointwise values of functions, define a *conjugation map* on V to be a complex-conjugate-linear automorphism $j : V \to V$ with $\langle jx, jy\rangle = \langle y, x\rangle$ and $j^2 = 1$.

A conjugation map is equivalent to a complex-linear isomorphism $\Lambda : V \to V^*$ of V with its complex-linear dual, via Riesz-Fischer, by

$$\Lambda(y)(x) \;=\; \langle x, jy\rangle \;=\; \overline{\langle y, jx\rangle}.$$

Assume j stabilizes D and that $T(jx) = jTx$ for $x \in D$. Then j respects $\langle,\rangle_1$:

$$\langle jx, jy\rangle_1 \;=\; \langle y, Tx\rangle \;=\; \langle y, x\rangle_1 \;\text{ for } x, y \in D.$$

Also, j commutes with T_{Fr}:

$$\langle x, T_{\mathrm{Fr}}^{-1} jy\rangle_1 \;=\; \langle x, jy\rangle \;=\; \langle y, jx\rangle \;=\; \langle T_{\mathrm{Fr}}^{-1} y, jx\rangle_1 \;=\; \langle x, jT_{\mathrm{Fr}}^{-1} y\rangle_1 \;\; (\text{for } x \in V_1, y \in V).$$

Let V_{-1} be the complex-linear dual of V_1, not identified with V_1. The inner product $\langle,\rangle_{-1}$ on V_{-1} comes via polarization from the norm $|\lambda|_{-1} \;=\; \sup_{x\in V_1 : |x|_1 \le 1} |\lambda x|$.
We have $V_1 \subset V \subset V_{-1}$.

Extending T_{Fr} to V_1 By design, $T : D \to V \subset V_{-1}$ is continuous when V has the subspace topology from V_{-1}:

$$|Ty|_{-1} \;=\; \sup_{|x|_1 \le 1} |\Lambda(Ty)(x)| \;=\; \sup|\langle x, jTy\rangle| = |\langle x, Tjy\rangle| \le \; \sup|x_1| \cdot |y_1| \;=\; |y|_1.$$

by Cauchy-Schwarz-Bunyakowsky. Thus the map $T : D \to V$ extends by continuity to an everywhere-defined, continuous map $T_{\mathrm{bd}} : V_1 \to V_{-1}$ by

$$(T_{\mathrm{bd}} y)(x) = \langle x, jy\rangle_1.$$

Further, $T_{\mathrm{bd}} : V_1 \to V_{-1}$ agrees with $T_{\mathrm{Fr}} : D_1 \to V$ on the domain $D_1 = BV$ of T_{Fr}, since $(T_{\mathrm{bd}}y)(x) = \langle x, jy\rangle_1 = \langle Tx, jy\rangle = \langle Tx, T_{\mathrm{Fr}}^{-1}T_{\mathrm{Fr}}jy\rangle = \langle T_{\mathrm{Fr}}^{-1}Tx, T_{\mathrm{Fr}}jy\rangle$

$$= \langle x, T_{\mathrm{Fr}}jy\rangle = \Lambda(T_{\mathrm{Fr}}y)(x) \quad \text{for } x \in D \text{ and } y \in D_1.$$

This follows since T_{Fr} extends T, and noting the density of D in V.

Claim: The domain of T_{Fr} is $D_1 = \{u \in V_1 : T_{\mathrm{bd}}u \in V\}$

Proof: $T_{\mathrm{bd}}u = f \in V$ implies that

$$\langle x, ju\rangle_1 = (T_{\mathrm{bd}}u)(x) = \Lambda(T_{\mathrm{bd}}u)(x) = \Lambda(f)(x) = \langle x, jf\rangle \text{ for all } x \in V_1.$$

By the characterization of the Friedrichs extension, $T_{\mathrm{Fr}}(ju) = jf$. Since T_{Fr} commutes with j, we have $T_{\mathrm{Fr}}u = f$.

4.3 Friedrichs extensions of restrictions

Extend the complex conjugation j to V_{-1} by $(j\lambda)(x) = \overline{\lambda(jx)}$ for $x \in V_1$, and write

$$\langle v, \theta\rangle_{V_1\times V_{-1}} = (j\theta)(x) = \overline{\theta(jx)} \text{ (for } x \in V_1 \text{ and } \theta \in V_{-1}).$$

For $\theta \in V_{-1}$,

$$\theta^{\perp} = \{x \in V_1 : \langle x, \theta\rangle_{V_1\times V_{-1}}\}.$$

is a closed co-dimension-one subspace of V_1 in the $\langle , \rangle_1$-topology.

Assume $\theta \notin V$. This implies density of $\theta^{\perp}$ in V in the $\langle , \rangle$-topology.

Claim: The Friedrichs extension $T_\theta = (T|_{\theta^\perp})_{\mathrm{Fr}}$ of the restriction $T|_{\theta^\perp}$ of T to $D \cap \theta^\perp$ ignores θ, in the sense that $T_\theta u = f$ for $u \in V_1$ and $f \in V$ exactly when $T_{\mathrm{bd}}u = f + c\theta$ for some $c \in \mathbb{C}$. Letting D_1 be the domain of T_{Fr}, the domain of T_θ is

$$\text{domain } T_\theta = \{x \in V_1 : \langle x, \theta\rangle_{1\times V_{-1}} = 0,\ T_{\mathrm{bd}}x \in V + \mathbb{C}\cdot\theta\}.$$

Proof: $T_{\mathrm{bd}}u = f + c\cdot\theta$ is equivalent to

$$\langle x, ju\rangle_1 = T_{\mathrm{bd}}(u)(x) = (f + c\cdot\theta)(x) = \langle x, jf\rangle \text{ (for all } x \in \theta^\perp).$$

This gives $\langle x, ju\rangle_1 = \langle x, jf\rangle$. The topology on $\theta^\perp$ is the restriction of the $\langle , \rangle_1$-topology of V_1, while $\theta^\perp$ is dense in V in the $\langle , \rangle$-topology. Thus, $ju = T_\theta^{-1}jf$ by the characterization of the Friedrichs extension of $T_{\theta^\perp}$. Then $u = T_\theta^{-1}f$ since j commutes with T.

5 Meromorphic continuation of Eisenstein series

We review [Garrett 2011a], [Rankin-Selberg 1939], [Godement 1966], and [Colin de Verdiere 1980]. The quotient $\Gamma\backslash\mathfrak{H}$ is the union of a compact part and a geometrically trivial non-compact part:

$$\Gamma\backslash\mathfrak{H} = X_{\text{cpt}} \cup X_\infty.$$

where

$$X_\infty = \text{image of } \{x + iy \ : \ y \geq y_0\} \ \approx \ \text{circle} \ \times \ \text{ray}.$$

Define a smooth cut-off function τ as follows: fix $b < b'$ large enough so that the image of $\{z \in \mathfrak{H} : y > b\}$ in the quotient is in X_∞, let

$$\tau(y) = \begin{cases} 1 & \text{for } y > b' \\ 0 & \text{for } y < b \end{cases}.$$

With $\Gamma_\infty = SL_2(\mathbb{Z}) \cap P$, form a pseudo-Eisenstein series h_s by automorphizing the smoothly cut-off function $\tau(\text{Im}(z)) \cdot y^s$:

$$h_s(z) = \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \tau(\text{Im}(\gamma z)) \cdot \text{Im}(\gamma z)^s.$$

Since τ is supported on $y \geq b$ for large b, for any $z \in \mathfrak{H}$ there is at most one non-vanishing summand in the expression for h_s, and convergence is not an issue. Therefore, the pseudo-Eisenstein series is entire as a function-valued function of s. Let

$$\tilde{E}_s(z) = h_s - (\tilde{\Delta} - \lambda)^{-1}(\Delta - \lambda)h_s \quad (\text{where } \lambda = s(s-1)).$$

Theorem: With $\lambda = s(s-1)$ not non-positive real, $u = \tilde{E}_s - h_s$ is the unique element of the domain of $\tilde{\Delta}$ such that

$$(\tilde{\Delta} - \lambda)u = -(\Delta - \lambda)h_s.$$

Thus $\tilde{E}_s$ is the usual Eisenstein series E_s for $\text{Re}(s) > 1$, and gives an analytic continuation of E_s to $\text{Re}(s) > \frac{1}{2}$ with $s \notin (\frac{1}{2}, 1]$.

We will show, following [Garrett 2011a] that suitable restrictions Δ_a of Δ to subspaces of $L^2(\Gamma\backslash\mathfrak{H})$, where the constant term $\int_0^1 f(x+iy)dx$ vanishes above a fixed height $y = a$, denoted by $L^2(\Gamma\backslash\mathfrak{H})_a$ have Friedrichs extensions with *compact resolvents* and this gives meromorphic continuation of Eisenstein series to the whole plane.

Proposition: Test functions with constant term vanishing above height a are dense in $L^2(\Gamma\backslash\mathfrak{H})_a$. That is, $L^2(\Gamma\backslash\mathfrak{H})_a$ is the $L^2(\Gamma\backslash\mathfrak{H})$-closure of $(\mathrm{L}^2(\Gamma\backslash\mathfrak{H})_a \cap C_c^\infty(\Gamma\backslash\mathfrak{H}))$.

Let $C_c^\infty(\Gamma\backslash\mathfrak{H}) = (\mathrm{L}^2(\Gamma\backslash\mathfrak{H})_a \cap C_c^\infty(\Gamma\backslash\mathfrak{H}))$. Let Δ_a be the unbounded operator on $L^2(\Gamma\backslash\mathfrak{H})_a$ defined by restricting the domain of Δ to $C_c^\infty(\Gamma\backslash\mathfrak{H})$. The density of test functions in $L^2(\Gamma\backslash\mathfrak{H})_a$ proves the symmetry of Δ_a, extending integration by parts on test functions. Let $\tilde{\Delta}_a$ be the Friedrich extension of Δ_a to a self-adjoint unbounded operator on

$L^2(\Gamma\backslash\mathfrak{H})_a$. Let $\text{Sob}(+1)_a$ be the completion of $L^2(\Gamma\backslash\mathfrak{H})_a \cap C_c^\infty(\Gamma\backslash\mathfrak{H})$ with the Sob(+1)-topology, and similarly for $\text{Sob}(+2)_a$. Friedrichs' construction has the property

$$\text{Sob}(+2)_a \subset \text{domain } \tilde{\Delta}_a \subset \text{Sob}(+1)_a.$$

Now, let T_a be the distribution on $\Gamma\backslash\mathfrak{H}$ given by

$$T_a(f) = (c_P f)(ia) \quad \text{for } f \in C_c^\infty(\Gamma\backslash\mathfrak{H})_a.$$

T_a is a continuous linear functional on Sob(+1). Let $\mathfrak{A}$ be the distributions on $(0,\infty)$ supported at a, and understand by $\mathfrak{A} \circ c_P$ the composition of the constant-term map with distributions on $(0,\infty)$ supported on a.

Lemma The domain in $L^2\Gamma\backslash\mathfrak{H})$ of $\tilde{\Delta}_a$ is

$$\{f \in L^2(\Gamma\backslash\mathfrak{H})_a \ : \ \Delta f \in L^2(\Gamma\backslash\mathfrak{H}) + \mathfrak{A} \circ c_P\}.$$

The extension $\tilde{\Delta}_a$ is

$$\tilde{\Delta}_a f = g \qquad (\text{for } \Delta f \in g + \mathfrak{A} \circ c_P \text{ with } g \in L^2(\Gamma\backslash\mathfrak{H})_a).$$

Claim: The inclusion $\text{Sob}(+1)_a \to L^2(\Gamma\backslash H)_a$, from $\text{Sob}(+1)_a$ with its finer topology, is *compact*.

Proof: The total boundedness criterion for relative compactness requires that, given $\epsilon > 0$, the image of the unit ball B in $\text{Sob}(+1)_a$ in $L^2(\Gamma\backslash H)_a$ can be covered by finitely many balls of radius ϵ. The usual Rellich lemma reduces the issue to an estimate on the tail.

Given $c \geq a$, cover the image Y_0 of $\frac{\sqrt{3}}{2} \leq y \leq c+1$ in $\Gamma\backslash\mathfrak{H}$ by small coordinate patches U_i, and one large U_∞ covering the image Y_∞ in $y \geq c$. We invoke the compactness of Y_0 to obtain a finite subcover of Y_0. Choose a smooth partition of unity $\{\varphi_i\}$ subordinate to the finite subcover along with U_∞, letting φ_∞ be a smooth function that is identically 1 for $y \geq c$. A function f in the +1-index Sobolev space on Y_0 ia a finite sum of functions $\varphi \cdot f$. The latter can be viewed as having compact support on small opens in $\mathbb{R}^2$, thus identified ith functions on products of circles, and lying in the Sobolev +1-spaces there. Apply the Rellich compactness lemma to each of the finitely-many inclusion maps of Sobolev +1-spaces on the product of circles. Thus, $\varphi \cdot B$ is totally bounded in $L^2(\Gamma\backslash\mathfrak{H})$.

Therefore, to prove compactness of the global inclusion, it suffices to prove that, given $\epsilon > 0$, the cut-off c can be made sufficiently large so that $\varphi \cdot B$ is in a single ball of radius ϵ inside $L^2(\Gamma\backslash\mathfrak{H})$. It suffices to show that

$$\lim_{c\to\infty} \int_{y>c} |f(z)|^2 \frac{dx\, dy}{y^2} \to 0.$$

Denote by $\hat{f}(n)$ the Fourier coefficients of f. Take $c > a$ so that the 0^{th} Fourier coefficient $\hat{f}(0)$ vanishes identically. By Plancherel for the Fourier expansion in x, and then elementary inequalities: integrating over the part of Y_∞ above $y = c$, letting F be the Fourier transform in x,

$$\int\int_{>c} |f|^2 \frac{dxdy}{y^2} \;\le \frac{1}{c^2} \sum_{n\neq 0} \int_{y>c} |\hat{f}(n)|^2 \, dy \;\le \frac{1}{c^2}.$$

This uniform bound completes the proof that the image of the unit ball in $Sob(+1)_a$ in $L^2(\Gamma\backslash\mathfrak{H})_a$ is totally bounded. Therefore, the inclusion is indeed a compact map.

Corollary: For λ off a discrete set of points in $\mathbb{C}$, $\tilde{\Delta}_a$ has compact resolvent $(\tilde{\Delta}_a - \lambda)^{-1}$, and the parametrized family of compact operators

$$(\tilde{\Delta}_a - \lambda)^{-1} : L^2(\Gamma\backslash\mathfrak{H})_a \to L^2(\Gamma\backslash\mathfrak{H})_a.$$

is meromorphic in $\lambda \in \mathbb{C}$.

Setting

$$\tilde{E}_{a,s}(z) = h_s - (\tilde{\Delta}_a - \lambda)^{-1}(\Delta - \lambda)h_s \quad \text{(where } \lambda = s(s-1)\text{)}.$$

For $\lambda = s(s-1)$ not a non-positive real, $(\tilde{\Delta}_a - \lambda)^{-1}$ is a bijection of $L^2(\Gamma\backslash\mathfrak{H})_a$ to the domain of $\tilde{\Delta}_a$, so u $= \tilde{E}_{a,s}(z) - h_s$ is the unique element of the domain of $\tilde{\Delta}_a$, satisfying

$$(\tilde{\Delta}_a - \lambda)u \;\; = \;\; -(\Delta - \lambda)h_s.$$

Since the pseudo-Eisenstein series h_s is entire, the meromorphy of the resolvent $(\tilde{\Delta_a} - \lambda)^{-1}$ yields the meromorphy of $\tilde{E}_s(z)$. This claim proves that the space of square-integrable L^2 cuspforms on $\Gamma\backslash\mathfrak{H}$ has a Hilbert space basis of eigenfunctions for Δ. Also, $\tilde{\Delta}_a$ has compact resolvent, so has discrete spectrum. $\tilde{\Delta}_a$ has more genuine eigenfunctions than Δ, because certain truncated Eisenstein series (which are *not* eigenfunctions for Δ) are now eigenfunctions for $\tilde{\Delta}_a$. See [Lax-Phillips 1976].

6 Standard Estimates

6.1 We review [Garrett 2010]. For $G = SL_2(\mathbb{R})$, $\Gamma = SL_2(\mathbb{Z})$, and $K = SO(2)$, we prove the *standard estimate*

$$\sum_{|s_F|\le T} |F(g)|^2 + \frac{1}{2\pi}\int_{-T}^{T} |E_{\frac{1}{2}+it(g)}|^2 \, dt \ll_C \; T^2.$$

for cuspforms F with eigenvalues $\lambda_F = s_F(s_F - 1)$ for the Laplacian Δ, and Eisenstein series E_s.

We consider integral operators attached to compactly supported measures η on G and exploit the intrinsic sense of such operators on any locally-convex, quasi-complete G-representation space. For a representation π, V of G, and a compactly supported measure η, the action is

$$\eta \cdot \mathrm{v} \;=\; \int_G \pi(g)(\mathrm{v}) \, d\eta(g).$$

The theory of Gelfand-Pettis integrals assures the reasonable behavior of such integrals. These extend the definition of integral to continuous, compactly-supported vector-valued functions on finite measure spaces, with values in a quasi-complete, locally convex topological vector space.

A waveform f is the unique spherical vector in the copy of the unramified principal series (see below) representation it generates, up to a constant. Thus, for any left and right K-invariant compactly-supported measure η, the integral operator action

$$(\eta \cdot f)(x) \ = \ \int_G \pi(y)\, f(xy)\, d\eta(y).$$

produces another right K-invariant vector in the representation space of f. Necessarily, $\eta \cdot f$ is a scalar multiple of f. Let $\lambda_f(\eta)$ denote the eigenvalue

$$\eta \cdot f \ = \ \lambda_f \cdot f.$$

This is an intrinsinc representation theoretic relation, so the scalar $\lambda_f(\eta)$ can be computed in any model of the representation. We choose an umramified principal series

$$I_s \ = \ \{\text{smooth } K - \text{finite } \varphi \ : \ \varphi(\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \cdot g) \ = \ |\tfrac{a}{d}|^{2s} \cdot \varphi(g))\} \text{ (with } s \in \mathbb{C})$$

On I_s, the Laplacian has eigenvalue $\lambda_s = s(s-1)$.

6.2 Choice of an integral operator

Let $||g||$ be the square of the operator norm on G for the standard representation of G on $\mathbb{R}^2$ by matrix multiplication. In a Cartan decomposition,

$$||k_1 \cdot \begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{\frac{-r}{2}} \end{pmatrix} \cdot k_2|| \ = \ e^r \quad (\text{with } k_1, k_2 \in K, r \geq 0).$$

This norm gives a left G-invariant metric $d(,)$ on G/K by

$$d(gK, hK) \ = \ \log||h^{-1}g||.$$

The triangle inequality follows from the submultiplicativity of the norm.

Take η to be the characteristic function of the left and right K-invariant set of group elements of norm at most e^δ, with small $\delta > 0$. That is,

$$\eta(g) = \begin{cases} 1 & \text{for } ||g|| \leq e^\delta \\ 0 & \text{for } ||g|| > e^\delta \end{cases}.$$

6.3 Upper bound on a kernel

The map $f \to (\eta \cdot f)(x)$ on automorphic forms f can be expressed as integration of f against a sort of automorphic form q_x by winding up the integral, as follows.

$$(\eta \cdot f)(x) = \int_{\Gamma \backslash G} f(y) \cdot \sum_{\gamma \in \Gamma} \eta(x^{-1}\gamma y)\, dy.$$

Thus, for $x, y \in G$ put

$$q_x(y) \;=\; \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y).$$

The norm-squared of q_x, as a function of y alone, is

$$|q_x|^2_{L^2(\Gamma\backslash G)} \;=\; \int_G \sum_{\gamma\in\Gamma} \eta(x^{-1}\gamma y)\;\overline{\eta}(x^{-1}y)\;dy.$$

after unwinding. For both $\eta(x^{-1}\gamma y)$ and $\overline{\eta}(x^{-1}y)$ to be nonzero, the distance from x to both y and γy must be at most δ. By the triangle inequality, the distance from y to γy must be at most 2δ. For x in a fixed compact C, this requires that y be in the ball of radius δ, and that $\gamma y = y$. Since K is compact and Γ is discrete, the isotropy groups of all points in G/K are finite. Thus,

$$|q_x|^2_{L^2(\Gamma\backslash G)} \;\ll\; \int_{d(x,y)\leq\delta} 1\;dy \approx\; \delta^2.$$

6.4 Lower bound on eigenvalues

A non-trivial lower bound for $\lambda_f(\eta)$ can be given for $\delta \ll \frac{1}{t_f}$, as follows. With spherical function φ^o in the s^{th} principal series, the corresponding eigenvalue is

$$\lambda_s(\eta) \;=\; \int_G \eta(g)\varphi^o(g)\;dg \;=\; \int_{r\leq\delta} \varphi^o(k\cdot\begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{\frac{-r}{2}} \end{pmatrix})\;dg.$$

We need qualitative metrical properties of the Iwasawa decomposition. Let P^+ be the upper-triangular matrices in G with positive real entries, and $K = SO(2)$. Let $g \to p_g k_g$ be the decomposition. We claim that $\|g\| \leq \delta$ implies $\|p_g\| \ll\; \delta$ for small $\delta > 0$. This is immediate, since the Jacobian of the map $P^+ \to G/K$ near $e \in P$ is invertible.

But, also, the Iwasawa decomposition is easily computed here, and the integral expressing the eigenvalue can be estimated explicitly: elements of K can be parametrized as

$$k = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

and let $a = e^{\frac{r}{2}}$. Then

$$k\cdot\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;=\; \begin{pmatrix} * & * \\ -a\sin\theta & \frac{\cos\theta}{a} \end{pmatrix}.$$

Right multiplication by a suitable element k_2 of $SO(2)$ rotates the bottom row to put the matrix into P^+:

$$k\cdot\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\cdot k_2 \;=\; \begin{pmatrix} * & * \\ 0 & \sqrt{(-a|\sin\theta|)^2 + (\frac{\cos\theta}{a})^2} \end{pmatrix}.$$

Thus,

$$\lambda_s(\eta) \;\;=\;\; \int_{r\leq\,\delta} ((-a|\sin\theta|)^2 + (\frac{\cos\theta}{a})^2)^{-s}\;dg.$$

Rather than compute the integral exactly, make δ small enough to give a lower bound on the integrand, such as would arise from

$$|((-a|\sin\theta)^2 + (\frac{\cos\theta}{a})^2)^{-s} - 1| \ < \ \frac{1}{2}.$$

Therefore, for small r,

$$(e^{-\frac{r}{2}}\sin\theta)^2 + (\cos\theta e^{\frac{r}{2}})^2 \ = \ e^r \sin^2\theta + \frac{\cos^2\theta}{e^r} \ \approx \ (1+r)\sin^2\theta + (1-r)\cos^2\theta \ \ll \ 1+r.$$

Thus, for small $0 \le r \le \delta$,

$$|(e^r \sin^2\theta \ + \ \frac{\cos^2\theta}{e^r})^{-s} \ - \ 1| \ \ll \ |s| \cdot r.$$

Thus $0 \le r \le \delta \ll \frac{1}{s}$ suffices to make this less than $\frac{1}{2}$.. That is, with η the characteristic function of the δ-ball,

$$|\lambda_s(\eta)| \ = \ \int_G \eta(g)\varphi^o(g)\ dg \ \gg \ \int_{r \le \delta} 1 \ = \ \text{vol}\ (\delta\text{-ball}) \ \approx \ \delta^2.$$

for η the characteristic function of the δ-ball and for $|s| << \frac{1}{\delta}$. Taking δ as large as possible compatible with $\delta \ \ll \ \frac{1}{s}$ gives the bound

$$\lambda_s(\eta) \ \gg \ \delta^2 \quad (\text{for } |s| \ \ll \ \frac{1}{\delta}\ , \eta \text{ the characteristic function of the } \delta\text{-ball}).$$

From the L^2 automorphic spectral expansion of q_x, apply Plancherel

$$\sum_F |\langle q_x, F\rangle|^2 \ + \ \frac{|\langle q_x, q\rangle|^2}{\langle 1, 1\rangle} \ + \ \frac{1}{2\pi}\int_{-\infty}^{\infty} |\langle q_x, E_s\rangle|^2\ dt \ = \ |q_x|^2_{L^2(\Gamma\backslash G/K)} \ \ll \ \delta^2.$$

Truncating this to Bessel's inequality and dropping the single residual term yields

$$\sum_{s_F \le T} |\langle q_x, F\rangle|^2 \ + \ \frac{1}{2\pi}\int_{-T}^{T} |\langle q_x, E_s\rangle|^2\ dt \ \ll \ \delta^2.$$

Recall that for the spherical vector $f \in I_s$,

$$\langle q_x, f\rangle \ = \ \lambda_s(\eta) \cdot f.$$

and using the inequality $\lambda_s(\eta) \ \gg \ \delta^2$ for this restricted parameter range gives

$$\sum_{s_F \le T} (\delta^2 \cdot |F(x)|)^2 \ + \ \int_{-T}^{+T} (\delta^2 \cdot |E_s(x)|)^2\ dt \ \ll \ \delta^2.$$

Multiply through by $T^4 \ \approx \ \frac{1}{\delta^4}$ to obtain the **standard estimate**

$$\sum_{s_F \ll T} |F(x)|^2 \ + \ \int_{-T}^{T} |E_s(x)|^2\ dt \ \ll \ T^2.$$

These standard estimates give a sharp estimated for Levi-Sobolev spaces where various distribution lie.//

7 Higher rank spectral theory

We follow [DeCelles 2010], [Langlands 1976], and [Moeglin-Waldspurger 1989]. Given a parabolic P in $G = GL(3)$, and a function f on $Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}}$, recall the constant term of f along P is

$$c_P f(g) \;=\; \int_{N_k\backslash N_{\mathbb{A}}} f(ng)\, dn.$$

where N is the unipotent radical of P. An automorphic form satisfies the Gelfand condition if, for all maximal parabolics P, the constant term along P is zero. If f is also Z-finite and K-finite, it is called a cuspform.

Since the right action of G commutes with taking constant terms, the space of functions satisfying the Gelfand condition is G-stable, so is a subrepresentation. Gelfand and Pietesky-Shapiro showed that integral operators on this space are compact, so by the spectral theorem, this subrepresentation decomposes into a direct sum of irreducibles, each having finite multiplicity. We now decompose the rest of the L^2 space.

To obtain the L^2 decomposition of the non-cuspidal automorphic forms, we classify them according to their *cuspidal support*, i.e. the smallest parabolic on which they have a non-zero constant term. In GL_3, there are three association classes of proper parabolics. We will consider the standard parabolic subgroups: $P^3 = GL_3$, $P^{2,1}$ and $P^{1,2}$ the maximal parabolics, and $P^{1,1,1}$ the minimal parabolic, contained in both $P^{2,1}$ and $P^{1,2}$.

Observe that an automorphic form whose constant term along $P^3 = GL_3$ is zero is identically zero, and an automorphic form with cuspidal support P^3 is precisely a nonzero cusp form.

Consider an automorphic form f with cuspidal support $P^{2,1}$ and let $F = c_{2,1}f$. Then F is a non-zero left $N^{2,1}$-invariant function. So if it is spherical, it can be considered as a GL_2 automorphic form. In fact, it is a GL_2 cusp form, since the constant term of f along the minimal parabolic is zero.

Lastly, we have the automorphic forms whose cuspidal support is the minimal parabolic, i.e. those whose constant term along $P^{1,1,1}$ is nonzero.

While classifying automorphic forms according to cuspidal support is helpful, it does not give a concrete or explicit description of the various classes of automorphic forms. Recall from the SL_2 case that pseudo-Eisenstein series provided an explicit description of automorphic forms with cuspidal support P, and the space spanned by pseudo-Eisenstein series was the orthogonal complement to the space of cusp forms. In GL_3 things are more complicated, since there are more parabolic subgroups, but pseudo-Eisenstein series are still used to describe the orthogonal complement to the space of cusp forms.

Define pseudo-Eisenstein series

$$\Psi_\varphi(g) \quad = \quad \sum_{\gamma \in P_k \backslash G_k} \varphi(\gamma \cdot g).$$

where φ is a continuous, compactly supported function on $Z_{\mathbb{A}} N_{\mathbb{A}} M_k \backslash G_{\mathbb{A}}$. In GL_3, there are three different kinds of pseudo-Eisenstein series, corresponding to the three standard parabolic subgroups. Pseudo-Eisenstein series span the orthogonal complement to the space of cusp forms, and we will determine which pseudo-Eisenstein series span the complement. The key to proving orthogonality lies in the following

Claim: For any square-integrable automorphic form f, and any pseudo-Eisenstein series Ψ_φ^P, with P a parabolic subgroup,

$$\langle f, \Psi_\varphi^P \rangle_{Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}} \quad = \quad \langle c_P f, \varphi \rangle_{Z_{\mathbb{A}} N_{\mathbb{A}}^P M_k^P \backslash G_{\mathbb{A}}}.$$

From this adjointness, an L^2 automorphic form is a cusp form if and only if it is orthogonal to *all* pseudo-Eisenstein series. That is, the orthogonal complement to cusp forms is spanned by pseudo-Eisenstein series.

This relation allows us to decompose the space spanned by pseudo-Eisenstein series into orthogonal subspaces. In particular, if f is in the space spanned by pseudo-Eisenstein series, then it follows from the adjointness relation that f has cuspidal support $P^{2,1}$ or $P^{1,2}$ if and only if it is orthogonal to all $P^{1,1,1}$ pseudo-Eisenstein series. So the orthogonal complement to cuspforms decomposes into two orthogonal subspaces: the space spanned by $P^{1,1,1}$ pseudo-Eisenstein series, and the space of automorphic forms with cuspidal support $P^{2,1}$ or $P^{1,2}$.

We have to determine which pseudo-Eisenstein series are in the second subspace. A $P^{2,1}$ or $P^{1,2}$ pseudo-Eisenstein series with cuspidal data has cuspidal support $P^{2,1}$ or $P^{1,2}$. Any other $P^{2,1}$ or $P^{1,2}$ pseudo-Eisenstein series can be written as the sum of a $P^{1,1,1}$ pseudo-Eisenstein series and a $P^{2,1}$ or $P^{1,2}$ pseudo-Eisenstein series with cuspidal data. Therefore, the subspace consisting of automorphic forms with cuspidal data $P^{2,1}$ or $P^{1,2}$ is spanned by $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series with cuspidal data.

The space generateed by $P^{2,1}$ pseudo-Eisenstein series is actually the same as the space generated by $P^{1,2}$ pseudo-Eisenstein series. This is an example of a more general phenomenon: pseudo-Eisenstein series of associate parabolics span the same space.

So we have the following decomposition of $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})$ into orthogonal subspaces:

$$L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}}) \quad = \quad \text{(cfms)} \oplus (\text{span of } P^{1,1,1} \text{ ps-Eis}) \oplus (\text{span of } P^{2,1} \text{ ps-Eis, cspdl data}).$$

Decomposing Pseudo-Eisenstein Series

While we have a fairly nice description of the non-cuspidal automorphic forms in $L^2(Z_{\mathbb{A}} G_k \backslash G_{\mathbb{A}})$ in terms of pseudo-Eisenstein series, we would prefer a decomposition in terms of irreducibles. Following the GL_2 case, we will decompose the pseudo-Eisenstein series into genuine Eisenstein series. Due to plurality of parabolics in GL_3, we have several kinds of Eisenstein series in GL_3. For a parabolic P, the P-Eisenstein series is

$$E_\lambda \;=\; \sum_{\gamma\in P_k\backslash G_k} f_\lambda(\gamma g).$$

where f_λ is a spherical vector in a representation λ of M^P, extended to a P representation by left N-invariance, and induced up to G.

The key to obtaining the spectral decomposition for GL_2 pseudo-Eisenstein series is that the Levi component is a product of copies of GL_1, allowing us to reduce to the spectral theory for GL_1. For GL_3 we are able to use a similar approach for minimal parabolic pseudo-Eisenstein series, again because the Levi component is a product of copies of GL_1. The same methods will not work for decomposing $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series, because in these cases, the Levi component contains a copy of GL_2.

So we turn our attention to the decomposition of the minimal parabolic pseudo-Eisenstein series. We will need the functional equation of the Eisenstein series. Because of the increase in dimension, the symmetry of the Eisenstein series is more complex. The Eisenstein series can no longer be parametrized by one complex number s, since the data f_λ for the Eisenstein series is on a product of three copies of GL_1. The symmetries of the Eisenstein series can be described in terms of the action of the Weyl group W on the standard maximal torus A, on its Lie Algebra $\mathfrak{a}$, and on $i\mathfrak{a}^*$.

For GL_n, the standard maximal torus A is the product of n copies of GL_1 and representations of A are products of representations of GL_1; in the umramified case, these representations are just $y \to y^{s_i}$ for complex s_i. The Weyl group W is the group of permutation matrices in GL_n. It acts on A by permuting the copies of GL_1, and it acts on the dual in the canonical way, permuting the s_i in the unramified case.

We now describe the constant term and the functional equations of the Eisenstein series. The constant term of the Eisenstein series (along the minimal parabolic) has the form

$$c_P(E_\lambda) \;=\; \sum_{w\in W} c_w(\lambda)\cdot w\lambda.$$

where $w\lambda$ is the image of λ under the action of w and $c_w(\lambda)$ is a constant depending on w and λ with $c_1(\lambda) = 1$. The Eisenstein series has functional equations

$$c_w(\lambda)\cdot E_\lambda \;=\; E_{w\lambda} \quad \text{for all } w \in W.$$

We start the decomposition Ψ_φ by using the spectral expansion of its data φ. Recall that φ is left $N_{\mathbb{A}}$-invariant, so it is essentially a function on the Levi component, which is a product of copies of $k^\times\backslash\mathbb{J}$. Fujisaki's lemma implies that this is the product of a ray with a compact abelian group. All of our characters are simply trivial on the non-archimedean part. Spectrally decomposing φ is a higher-dimensional version of Mellin inversion.

$$\varphi \;=\; \int \langle \varphi, \lambda \rangle \cdot \lambda \; d\lambda.$$

Winding up,

$$\Psi_\varphi(g) \;=\; \int_{i\mathfrak{a}^*} \langle \varphi, \lambda \rangle \cdot E_\lambda(g) \; d\lambda.$$

Note that in order for this to be valid, the parameters of λ must have $\mathrm{Re}(s_i) \gg 1$. However, in order to use the symmetries of the functional equations, we need the parameters to be on the critical line $\rho + i\mathfrak{a}$, where ρ is the half-sum of positive roots. In moving the contours, we pick up some residues, which are constants. Breaking up the dual space according to Weyl chambers and changing variables,

$$\Psi_\varphi \;-\; (\text{residues}) \;=\; \sum_{w \in W} \int_{\text{1st Weyl Chamber}} \langle \varphi, w\lambda \rangle \cdot E_{w\lambda}(g) \; d\lambda.$$

Now use the functional equations to write

$$\Psi_\varphi \;-\; (\text{residues}) \;=\; \sum_{w \in W} \int_{(1)} \langle \varphi, c_w(\lambda) w\lambda \rangle \cdot E_\lambda(g) \; d\lambda.$$

We recognize the constant term of the Eisenstein series, and apply the adjointness relation

$$\sum_{w \in W} \langle \varphi, c_w(\lambda) w\lambda \rangle \;=\; \langle \varphi, c_P E_\lambda \rangle \;=\; \langle \Psi_\varphi, E_\lambda \rangle.$$

Therefore,

$$\Psi_\varphi(g) \;=\; \int_{(1)} \langle \Psi_\varphi, E_\lambda \rangle \cdot E_\lambda(g) \; d\lambda \;+\; \text{residues}.$$

Our next goal is to show that the remaining automorphic forms, namely those with cuspidal support $P^{2,1}$ or $P^{1,2}$ can be written as superpositions of genuine $P^{2,1}$ Eisenstein series. To do this, it is enough to decompose $P^{2,1}$ and $P^{1,2}$ pseudo-Eisenstein series with cuspidal support. Let $P = P^{2,1}$ and $Q = P^{1,2}$.

We look at pseudo-Eisenstein series with cuspidal data. The data for a P pseudo-Eisenstein series is smooth, compactly-supported, and left $Z_{\mathbb{A}} M_k^P N_{\mathbb{A}}^P$-invariant. Assume that the data is spherical. This means that this function is determined by its behavior on $Z_{\mathbb{A}} M_k^P \backslash M_{\mathbb{A}}^P$. Since this is *not* a product of copies of GL_1, we can not use the GL_1 spectral theory to accomplish the decomposition. Instead, this quotient is isomorphic to $GL_2(k) \backslash GL_2(\mathbb{A})$, so we use GL_2 spectral theory. If η is the data for a $P^{2,1}$ pseudo-Eisenstein series Ψ_η, we can write η as a tensor product $f \otimes v$ on

$$Z_{GL_2(\mathbb{A})} GL_2(k) \backslash GL_2(\mathbb{A}) \cdot Z_{GL_2(k)} \backslash Z_{GL_2(\mathbb{A})}.$$

Saying that the data is cuspidal means that f is a cuspform. Similarly, the data $\varphi = \varphi_{F,s}$ for a $P^{2,1}$-Eisenstein series is the tensor product of a GL_2 cusp form F and a character λ_s on GL_1. We show that $\Psi_{f,\mu}$ is the superposition of Eisenstein series $E_{F,s}$ where F ranges over an orthonormal basis of cusp forms and s is on the vertical line.

Using the spectral expansions of f and μ,

$$\eta \;=\; f\otimes\mu \;=\; \Big(\sum_{\text{cfms } F}\langle f, F\rangle\cdot F\Big)\cdot\Big(\int_s \langle \mu, \lambda_s\rangle\cdot\lambda_s\, ds\Big) \;=\; \sum_{\text{cfms } F}\int_s \langle \eta_{f,\mu}, \varphi_{F,s}\rangle\cdot\varphi_{F,s}\, ds.$$

So the pseudo-Eisenstein series can be re-expressed as a superposition of Eisenstein series.

Constant Terms of $GL(3)$-Eisenstein series

We work out computations of $GL(3)$-Eisenstein series using the Bruhat decomposition of G. Recall the Bruhat decomposition of GL_n

$$G \;=\; \bigcup PwQ \;=\; \bigcup_{w\in(W\cap P)\backslash W/(W\cap Q)} PwQ.$$

where W is the Weyl group and P and Q are parabolics. To compute the constant term along P of a Q-Eisenstein series,

$$c_P(E^Q_\varphi)(g) \;=\; \int_{N^P_k\backslash N^P_A}\sum_{\gamma\in Q_k\backslash G_k/P_k}\sum_{\beta\in Q_k\backslash Q_k\gamma P_k}\varphi(\gamma\beta n g)\, dn$$

$$= \int_{Q_k\backslash G_k/Q_k}\int_{N^P_k\backslash N^P_{\mathbb{A}}}\sum_{\beta\in Q_k\backslash Q_k\gamma P_k}\sum_{\beta\in Q_k\backslash Q_k w P_k}\varphi(\gamma\beta n g)\, dn$$

$$= \sum_{w\in(W\cap P)\backslash W/(W\cap Q)}\int_{N^P_k\backslash N^P_{\mathbb{A}}}\sum_{\beta\in Q_k\backslash Q_k w P_k}\varphi(\beta n g)\, dn$$

$$= \sum_{w\in(W\cap P)\backslash W/(W\cap Q)}\int_{N^P_k\backslash N^P_{\mathbb{A}}}\sum_{\beta\in(w^{-1}Q_k w\cap P_k)\backslash P_k}\varphi(w\beta n g)\, dn.$$

Further computation is dependent on the choice of P and Q. For example, consider $P = Q = P^{1,1,1}$ the minimal parabolic. Then the constant term takes the form

$$c_{1,1,1}(E^{1,1,1}_\varphi) \;=\; \sum_{w\in W} c_w(\lambda)\, w\lambda \quad \text{where } c_1(\lambda) \;=\; 1.$$

when φ is in the principal series I_λ. We work out the computations.
The double coset space $(W\cap P)\backslash W/(W\cap P)$ is the entire Weyl group W, and since the Levi component is invariant under conjugation by elements of W, $PwP = PwN$ for all w. Therefore, the constant term is

$$c_{1,1,1}(E^{1,1,1}_\varphi)(g) \;=\; \sum_{w\in W}\int_{N_k\backslash N_{\mathbb{A}}}\sum_{\beta\in(w^{-1}P_k w\cap N_k)\backslash N_k}\varphi(w\beta n g)\, dn.$$

for $w = 1$,

$$\int_{N_k\backslash N_{\mathbb{A}}} \varphi(ng)\, dn \quad = \quad \varphi(g).$$

and for $w = w_0$, the long Weyl element, the intersection, $w_0^{-1}P_k w_0 \cap N_k$ is trivial, so there is unwinding

$$\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in N_k} \varphi(w_0\gamma ng)\, dn \quad = \quad \int_{N_k} \varphi(w_0 ng)\, dn.$$

and this integral factors over primes because φ does.

The integrals corresponding to the four other elements of the Weyl group have partial unwinding. Consider first $w = \sigma_0$, the element corresponding to the reflection of the first positive simple root. The quotient $(\sigma^{-1}N_k\sigma \cap N_k)\backslash N_k$ is isomorphic to the GL_2 unipotent radical, denoted $N^{1,1}$. Therefore the integral simplifies to

$$\int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma\in(\sigma^{-1}N_k\sigma\cap N_k)\backslash N_k} \varphi(\sigma\gamma ng)\, dn \quad = \quad \int_{N^{1,1}_{\mathbb{A}}} \varphi(\sigma ng)\, dn.$$

We can compute the terms corresponding to the other Weyl elements similarly. For $w = \tau$, the element corresponding to the reflection of the second positive simple root, we get

$$\int_{N^{1,1}_{\mathbb{A}}} \varphi(\tau ng)\, dn.$$

For $w = \tau\sigma$,

$$\int_{N^{2,1}_{\mathbb{A}}} \varphi(\tau\sigma ng)\, dn.$$

Finally, for $w = \sigma\tau$, we get

$$\int_{N^{1,2}_{\mathbb{A}}} \varphi(\sigma\tau ng)\, dn.$$

These integrals factor over primes, and the local integrals are intertwining operators among principal series: $T_{w,\lambda_v} : I_{\lambda_v} \to I_{w\lambda_v}$. For example, consider the local integral for $w = \sigma$. Using right K_v-invariance,

$$T_{w,\lambda_v}\varphi(g) \quad = \quad \int_{N_v} \varphi_v(\sigma ng)\, dn \quad = \quad \int_{N_v} \varphi(\sigma nn_g m_g)\, dn \quad = \quad \int_{N_v} \varphi_v(\sigma nm_g).$$

Changing variables $n \to m_g n m_g^{-1}$ and using the P-equivariance of φ_v by λ_v,

$$T_{w,\lambda_v}\varphi_v(g) \quad = \quad \delta \int_{N_v} \lambda_v(\sigma m_g\sigma^{-1})\varphi_v(\sigma n)\, dn.$$

This is the action of W on λ_v, so

$$T_{w,\lambda_v}\varphi_v(g) \quad = \quad \delta(m_g)\cdot\sigma\lambda_v(m_g)\int_{N_v} \varphi_v(\sigma n)\, dn \quad = \quad \delta(m_g)\cdot\sigma\lambda_v(m_g)\cdot T_{\sigma,\lambda_v}\varphi_v(1).$$

Therefore the constant term is

$$c_{1,1,1}(E_\varphi^{1,1,1})(g) \;=\; \sum_{w\in W}\Big(\prod_v T_{w,\lambda_v}\varphi(1)\Big)\cdot\delta(m_g)\cdot w\lambda(m_g).$$

Defining $c_w(\lambda)$ to be the constant in front and renormalizing to eliminate the modular function, we obtain the desired expression for the constant term:

$$c_{1,1,1}(E_\varphi^{1,1,1})(g) \;=\; \sum_{w\in W} c_w(\lambda)\cdot w\lambda(g).$$

Truncation and Maas-Selberg relations

We follow [Garrett 2011a]. Maas-Selberg relations allow us to compute inner products of *truncated Eisenstein series*. The crucial corollary is that Eisenstein series arising from self-associate parabolic subgroups have no poles in the right half-plane. Furthermore, any such pole is simple, and the residues are square-integrable. Let $G = GL_n(\mathbb{R})$, $\Gamma = GL_n(\mathbb{Z})$, and $K = O(n,\mathbb{R})$. For n_1 and n_2 positive integers such that $n_1 + n_2 = n$, define the corresponding standard maximal parabolic subgroup

$$P \;=\; \begin{pmatrix} n_1\times n_2 & * \\ 0 & n_2\times n_2 \end{pmatrix}.$$

with unipotent radical

$$N^P \;=\; \begin{pmatrix} 1_{n_1} & 0 \\ 0 & 1_{n_1} \end{pmatrix}.$$

and standard Levi component

$$M^P \;=\; GL_{n_1}\times GL_{n_2}.$$

Fix a standard parabolic P and N its unipotent radical. For f an $N_\mathbb{Z} = N\cap\Gamma$-invariant function, the constant term of f along the parabolic P is defined as usual to be

$$c_P f(g) \;=\; \int_{N_\mathbb{Z}\backslash N} f(ng)\,dn.$$

Fix integers n_1, n_2. For $i = 1,2$, let f_i be *cuspforms* on $GL(n_i,\mathbb{R})$. Let $P = P_{n_1,n_2}$, and put

$$\varphi(nmk) \;=\; \varphi_{s,f}(nmk) \;=\; |\det m_1|^{n_2 s}|\det m_2|^{-n_1 s} f_1(m_1) f_2(m_2).$$

where

$$m \;=\; \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}.$$

with $m_i \in GL(n_i)$, so that m is in the standard Levi component M of the parabolic subgroup P, $n\in N$ its unipotent radical, and $k\in K$. Let $P_\mathbb{Z} = \Gamma\cap P$. Define the associated Eisenstein series

$$E^P(\varphi)(g) \;=\; \sum_{\gamma\in P_\mathbb{Z}\backslash\Gamma}\varphi(\gamma g).$$

For Re(s) sufficiently positive, this series converges absolutely and uniformly on compacta. It is a left $GL(n_1+n_2,\mathbb{Z})$-invariant right $O(n_1+n_2)$-invariant function with trivial central character.

Now define the truncation operators. For a standard maximal proper parabolic subgroup $P = P_{n_1,n_2}$, for $g = nmk$ with $m \in M^P$, $n \in N^P$, and $k \in O(n)$, define

$$h^P(g) \quad = \quad \frac{|\det m_1|^{n_2}}{|\det m_2|^{n_1}} \quad = \quad \delta^P(nm) \quad = \quad \delta^P(m).$$

where δ^P is the modular function on P. For fixed large real T, the T-**tail** of the P-constant term of a left $N^P_{\mathbb{Z}}$-invariant function F is

$$c_P^T \quad = \quad \begin{cases} 0 & \text{if } h^P(g) \geq T \\ c_P f(y) & \text{if } h^P(g) < T \end{cases}.$$

We want the truncations of Eisenstein series to be in L^2, and also so that we can calculate their inner products reasonably. Also, there should be no obstacle to meromorphic continuation of the tail in the truncation.

Proposition: The truncated Eisenstein series $\Lambda^T E^P_\varphi$ is of rapid decay in all Siegel sets.

Theorem: (*Maas-Selberg relations*)

$$\langle \Lambda^T E^P_\varphi, \Lambda^T E^P_\phi \rangle = \langle f, h \rangle \frac{T^{s+\bar{r}-1}}{s+\bar{r}-1} + \langle f^w, h \rangle c^\phi_r \cdot \frac{T^{(1-s)+\bar{r}-1}}{(1-s)+\bar{r}-1} + \langle f, h^w \rangle \overline{c^\varphi_s} \cdot \frac{T^{s+(1-\bar{r})-1}}{s+(1-\bar{r})-1}$$

$$+ \langle f^w, h^w \rangle c^\varphi_s \overline{c^\phi_r} \cdot \frac{T^{(1-s)+(1-\bar{r})-1}}{(1-s)+(1-\bar{r})-1}.$$

Corollary: For maximal proper parabolics P in $GL(n)$, on the half-plane Re(s) $\geq \frac{1}{2}$ an Eisenstein series E^P_φ has no poles whatsoever if P is not self-associate. If P is self-associate, the only possible poles are on the real line, and only occur if $\langle f, f^w \rangle$ is not equal to zero. In that case, any pole is simple, and the residue is in L^2.

8 Future Work...

Speh forms are induced from cuspforms on SL_2, and though they are in L^2, they are not of rapid decay. Speh forms illustrate the complications of doing harmonic analysis on higher rank groups. Indeed, for SL_2, we saw that the residual spectrum consisted only of constants. For SL_4, there is a marked difference, in that Speh forms also enter into the residual spectrum. This provides an incentive for setting up a finer harmonic analysis on higher rank groups; in particular, one that doesn't use gritty details.

Recall that the theory of the constant term [Moeglin-Waldspurger 1995] asserts that the asymptotic behavior of a $\mathfrak{z}$-finite and K-finite automorphic form is dominated by the asymptotic behavior of its constant term. Determination of constant terms for $GL(n)$ Eisenstein series with $n > 2$ is less elementary than for $GL(2)$, and new phenomenon arise as well. To begin with, consider the minimal parabolic subgroup $P = Q = P^{1,1,1}$. The double coset space $(W \cap P)\backslash W/(W \cap P)$ is the entire Weyl group W. For $w = 1$, we get

$$\int_{N_k\backslash N_{\mathbb{A}}} \varphi(ng)\, dn \quad = \quad \varphi(g) \cdot \mathrm{vol}(N_k\backslash N_{\mathbb{A}}).$$

while the longest Weyl element w_0, gives the integral

$$\int_{N_{\mathbb{A}}} \varphi(w_0 ng)\, dn.$$

We want to compute the above integral. The integral factors over primes into a product of integrals of the form

$$\int_{N_v} \varphi_v(w_0 ng)\, dn.$$

The map sending φ to $\int_{N_v} \varphi_v(w_0 ng)\, dn$ is an intertwining operator among principal series, and these intertwining operators factor as intertwining operators corresponding to reflections in the Weyl group. The important idea (explained below) is that the expression of this intertwining operator as a composition of intertwining operators for simple reflections reduces the computation to that of the computation of the constant term for $GL(2)$ Eisenstein series. Furthermore, because a G-homomorphism maps a normalized spherical vector to a scalar multiple of the normalized vector in the target, we can identify that scalar unambiguously. That is, $T_{w_0,\lambda_v} : I_{\lambda_v} \to I_{w_0\lambda_v}$. This intertwining operator can be written as a composition of intertwining operators among principal series associated to simple reflections. Since the longest Weyl element factors as $w_0 = \sigma \cdot \tau \cdot \sigma$ where σ is the reflection corresponding to the first simple root and τ is the reflection corresponding to the second simple root. Rewrite

$$T_{w_0,\lambda_v} : I_{\lambda_v} \to I_{\sigma\lambda_v} \to I_{\tau\sigma\lambda_v} \to I_{\sigma\tau\sigma\lambda_v}.$$

Consider the first map in the above composition from $I_{\lambda_v} \to I_{\sigma\lambda_v}$. This intertwining is given by the integral

$$\int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma ng)\, dn.$$

where $N_{\mathbb{A}}^{1,1}$ denotes the GL_2 unipotent radical, that is all matrices of the form

$$\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad : \text{where } x \in \mathbb{R}\}.$$

To see why this is true, observe that

$$\int_{N_{\mathbb{A}}} \varphi(\sigma ng)\, dn \quad = \quad \int_{N_k\backslash N_{\mathbb{A}}} \sum_{\gamma \in (\sigma^{-1} N_k \sigma \cap N_k)\backslash N_k} \varphi(\sigma\gamma ng)\, dn.$$

this integral in turn is equal to

$$\int_{N_k^{1,1}\backslash N_{\mathbb{A}}^{1,1}} \int_{N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}} \sum_{\gamma \in N_k^{1,1}} \varphi(\sigma\gamma nug)\, du\, dn \quad = \quad \int_{N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1}} \int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma nug)\, dn\, du.$$

Reversing the order of integration shows that the above equals

$$\int_{N_{\mathbb{A}}^{1,1}} \int_{N_k^{2,1}\backslash N_{\mathbb{A}2,1}} \varphi(u\sigma ng)\, du\, dn \quad = \quad \mathrm{vol}(N_k^{2,1}\backslash N_{\mathbb{A}}^{2,1})\cdot \int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma ng)\, dn \quad = \quad \int_{N_{\mathbb{A}}^{1,1}} \varphi(\sigma ng)\, dn.$$

This gives

$$\int_{N_{\mathbb{A}}^{1,1}} \varphi_{(s_1,s_2,s_3)}(\sigma ng)\, dn.$$

where σ is the permutation matrix corresponding to the first simple reflection. Since principal series are generically irreducible, maps among them are completely determined by where they send the normalized spherical vector. This allows us (by slight abuse of notation of spherical vectors), reduce the calculation to a $GL(2)$ calculation. Indeed, the matrix for σ is given by

$$\sigma \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Writing $n = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we see that the only interaction in matrix multiplication taking place is in the upper 2 by 2 block of the matrix. Indeed,

$$\sigma \cdot n \quad = \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We must rewrite this as $p \cdot k$ where $p \in P$ and $k \in K$, so that we can apply the spherical vector to it. Observe that

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{x}{\sqrt{x^2+1}} & \frac{1}{\sqrt{x^2+1}} & 0 \\ \frac{-1}{\sqrt{x^2+1}} & \frac{x}{\sqrt{x^2+1}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad = \quad \begin{pmatrix} \frac{-1}{\sqrt{x^2+1}} & * & * \\ 0 & \sqrt{x^2+1} & * \\ 0 & 0 & 1 \end{pmatrix}.$$

Observe, however, that when we apply the spherical vector φ_s^{sph} to this matrix, we will get $(\frac{1}{x^2+1})^s$. If we are clairyvoyant (or have prior acquaintance with $GL2$ calculations), this is the same as $\int_{N_{\mathbb{A}}} \varphi(\sigma \cdot n)\, dn$, where φ is the $GL(2)$ spherical vector. All of the action really takes place in the upper 2×2 block of the matrix.

Therefore, the computation reduces to the $GL(2)$ calculation

$$\int_{N_{\mathbb{A}}} \varphi_{(s_1,s_2)}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ng)\, dn.$$

Here $\varphi_{(s_1,s_2)}$ is the standard spherical vector in the principal series I_s. That is,

$$\varphi_{(s_1,s_2)}(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;=\; |a|^{s_1} \cdot |d|^{s_2}).$$

where we insist that $s_1 + s_2 = 0$ for simplicity. Factor the integral into integrals taken over N_v for all places v. We set g equal to the identity. Write $n = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. Observe that $\sigma \cdot n = \begin{pmatrix} 0 & 1 \\ 1 & t \end{pmatrix}$. For archimedean places, multiplying by an appropriate element of K, and using right K-invariance of the spherical vector, we can transform this matrix to be $\begin{pmatrix} \frac{-1}{\sqrt{t^2+1}} & \frac{t}{\sqrt{t^2+1}} \\ 0 & \sqrt{t^2+1} \end{pmatrix}$. We can again multiply by an element of $GL_2(\mathbb{Z}_v)$ to get $\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}$. Applying φ_v to this matrix gives 1 provided $|t|_v \leq 1$ and $|t|_v^{-2s}$ for $|t|_v > 1$.

We recall the computation which allows us to find the constant term of the GL2 Eisenstein series. Parametrizing $P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}$ via the Bruhat decomposition, we get

$$\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} E_s(ng)\, dn \;=\; \int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \sum_{\gamma\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}} \varphi(\gamma ng)\, dn \;=\; \sum_{w\in P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/N_{\mathbb{Q}}} \int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \sum_{\gamma\in P_{\mathbb{Q}}\backslash P_{\mathbb{Q}}wN_{\mathbb{Q}}} \varphi(\gamma ng)\, dn.$$

By the Bruhat decomposition, $P_{\mathbb{Q}}\backslash G_{\mathbb{Q}}/N_{\mathbb{Q}}$ has exactly two representatives, 1, and $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so the constant term reduces to

$$\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \varphi(ng)\, dn \;+\; \int_{N_{\mathbb{A}}} \varphi(wng)\, dn.$$

By the left $N_{\mathbb{A}}$-invariance of φ, the first of the two summands is

$$\int_{N_{\mathbb{Q}}\backslash N_{\mathbb{A}}} \varphi(ng)\, dn \;=\; \varphi(g) \cdot \mathrm{vol}(N_{\mathbb{Q}}\backslash N_{\mathbb{A}}).$$

Since the integral in the second summand unwound, it factors over primes

$$\int_{N_{\mathbb{A}}} \varphi(wng)\, dn \;=\; \prod_{v\leq\infty} \int_{N_v} \varphi(wng)\, dn.$$

The v-adic local factor is

$$\int_{|t|_v\leq 1} 1\, dt \;+\; \int_{|t|_v>1} |t|_v^{-2s}\, dt = \;=\; 1+\sum_{l=1}^{\infty} |p^{-l}|_v^{-2s} \cdot \int_{p^{-l}\mathbb{Z}_p^\times} 1\, dt \;=\; 1+\sum_{l=1}^{\infty} (p^l)^{-2s} \cdot p^{l-1}(p-1).$$

This in turn is equal to

$$= \;1 + (1 - \frac{1}{p})\frac{p^{1-2s}}{1 - p^{1-2s}} \;=\; \frac{1 - p^{-2s}}{1 - p^{1-2s}} \;=\; \frac{\zeta_v(2s-1)}{\zeta_v(2s)}.$$

The product of all these local zeta functions is the completed zeta function $\frac{\zeta(2s-1)}{\zeta(2s)}$. Therefore, the finite-prime part of the big-cell summand is a quotient of zeta functions. The archimedean factor of the big-cell summand of the constant term is calculated to be $y^{1-s} \cdot \frac{\zeta_\infty(2s-1)}{\zeta_\infty(2s)}$. Therefore, with $\xi(s)$ the completed zeta function $\xi(s) = \zeta_\infty(s) \cdot \zeta(s)$, the constant term of E_s is

$$c_P E_s(x+iy) \quad = \quad y^s \quad + \quad \frac{\xi(2s-1)}{\xi(2s)} \cdot y^{1-s}.$$

Therefore,

$$\int_{N_{\mathbb{A}}} \varphi_{(s_1,s_2)}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ng)\, dn \quad = \quad \frac{\xi(s_1-s_2-1)}{\xi(s_1-s_2)}.$$

The next intertwining operator in the composition is $\tau : I_{\sigma\cdot\lambda} \to I_{\tau\cdot\sigma\cdot\lambda}$. We have to keep track of what happens to the spherical vector $\lambda_{(s_1,s_2,s_3)}$. The action of σ on λ_s isn't as simple as it seems. Indeed, writing the integral

$$\int_{k_v} \varphi_s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix})\, dx.$$

we see, using commutation relations that this integral can be rewritten effectively as

$$\int_{k_v} \varphi_s(\begin{pmatrix} a_2 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x\cdot\frac{a_2}{a_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})\, dx.$$

using the fact that the spherical vector is λ_s-invariant under P, we write the above as

$$\lambda_s\begin{pmatrix} a_2 & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_3 \end{pmatrix}\int_{k_v} \varphi_s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x\cdot\frac{a_2}{a_1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})\, dx.$$

which is

$$|a_1|^{s_2} \cdot |a_2|^{s_1} \cdot |a_3|^{s_3} \cdot |\frac{a_2}{a_1}| \int_{k_v} \varphi_s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})\, dx.$$

Finally, this gives

$$|a_1|^{s_2+1} \cdot |a_2|^{s_1-1} \cdot |a_3|^{s_3} \int_{k_v} \varphi_s(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})\, dx.$$

Therefore the "new" spherical vector is $\varphi_{(s_2+1,s_1-1,s_3)}$ and we can now compute the next intertwining operator $T_\tau : I_{\lambda_{(s_2,s_1,s_3)}} \to I_{\tau\cdot\lambda_{(s_2,s_3,s_1)}}$. The map T_τ is given by the integral

$$\int_{N_{\mathbb{A}}} \varphi_{(s_2+1,s_1-1,s_3)}(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix})\, dx.$$

Again the only action takes place in the lower left $GL(2)$ block, so that the integral reduces to

$$\int_{N_{\mathbb{A}}} \varphi_{(s_1-1,s_3)}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\, dx.$$

but we recognize this as $\frac{\xi(s_1-s_3-2)}{\xi(s_1-s_3-1)}$.

The next intertwing operator to be considered is $T_\sigma; I_{\tau\cdot\sigma\cdot\lambda_s} \to I_{\sigma\cdot\tau\cdot\sigma\cdot\lambda_s}$. Observe that the spherical vector in the first principal series is now given by $\varphi_{(s_2+1,s_3+1,s_1-2)}$. The relevant integral will therefore be

$$\int_{N_{\mathbb{A}}} \varphi_{(s_2+1,s_3+1,s_1-2)}(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})\, dx.$$

Again, the only interaction takes place in the upper left $GL(2)$ block, so we reduce to

$$\int_{N_{\mathbb{A}}} \varphi_{(s_2+1,s_3+1)}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix})\, dx.$$

By now we know that this integral is equal to $\frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)}$.
Therefore, the constant term of the minimal parabolic $GL(3)$ Eisenstein series is given by

$$c_{P^{1,1,1}} E_s(g) \quad = \quad \frac{\xi(s_1-s_2-1)}{\xi(s_1-s_2)} \cdot \frac{\xi(s_1-s_3-2)}{\xi(s_1-s_3-1)} \cdot \frac{\xi(s_2-s_3-1)}{\xi(s_2-s_3)}.$$

9 Appendix 1: Principal series

We use the basis $H, X, Y \in \mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$. The standard parabolic subgroup $P = NM$ of $G = SL_2(\mathbb{R})$ consists of upper-triangular matrices in G. The s^{th} unramified principal series representation I_s of $G = SL_2(\mathbb{R})$ induced from the character χ_s on P, is

$$I_s = \{f \in C_c^\infty(G) : f(pg) = \chi_s(p) \cdot f(g), \quad \text{for all } p \in P,\ g \in G\}.$$

The representation of the group G on I_s is by right translation,

$$g \cdot f(x) = f(xg) \ \ (\text{for } f \in I_s, \text{ and } x, g \in G).$$

The Lie algebra $g = \mathfrak{sl}_2(\mathbb{R})$ acts correspondingly,

$$\gamma \cdot f(x) = \frac{\partial}{\partial t}|_{t=0} f(xe^{t\gamma}) \quad (\text{ for } f \in I_s, x \in G, \text{and } \gamma \in \mathfrak{g}\ ).$$

The action of $U\mathfrak{g}$ and Ω is induced from the action of the Lie algebra. The G-invariance of Ω can be exploited: for $g \in G$,

$$(\Omega f)(g) = (\Omega f)(1 \cdot g) = (g \cdot (\Omega f))(1) = (\Omega \cdot (g \cdot f))(1).$$

Note that $F = F_g = g \cdot f$ is still in I_s. That is, $\Omega f(g) = (\Omega F)(1)$, using the G-invariance of Ω. Thus, for any basis x_i of $\mathfrak{g}$ and dual basis x_i^* relative to the trace form,

$$\Omega F(1) \;=\; \sum_i \frac{\partial}{\partial t'}\Big|_{t'=0} \frac{\partial}{\partial t}\Big|_{t=0} F(e^{t' x_i} e^{t x_i^*}).$$

In particular, taking the basis H, X, Y with $H^* = \frac{1}{2}H, X^* = Y, Y^* = X$,

$$F(e^{t' H} e^{tH^*}) \;=\; \chi_s(e^{t' H} \cdot F(e^{tH^*}) \;=\; \Big|\frac{e^{t'}}{e^{-t'}}\Big|^s \cdot F(e^{tH^*}).$$

Thus,

$$\frac{\partial}{\partial t'}\Big|_{t=0} F(e^{t' H} e^{tH^*}) \;=\; 2s \cdot F(e^{tH^*}).$$

and

$$HH^* F(1) \;=\; \frac{1}{2} H^2 F(1) \;=\; \frac{1}{2} \cdot (2s)^2 \cdot F(1).$$

Similarly,

$$F(e^{t' X} e^{tX^*}) \;=\; \chi_s(e^{t' X}) \cdot F(e^{tX^*}) \;=\; 1 \cdot F(e^{tX^*}).$$

so

$$\frac{\partial}{\partial t'}\Big|_{t=0} F(e^{t' X} e^{tX^*}) \;=\; 0.$$

This motivates the rearrangement

$$XY + YX \;=\; 2XY + [Y, X] \;=\; 2XY - H.$$

Thus,

$$(XX^* + YY^*)F(1) \;=\; (XY + YX)F(1) \;=\; (2XY - H)F(1) \;=\; 0 - (2s)(1).$$

Altogether,

$$\Omega F(1) \;=\; (\frac{1}{2} H^2 + XY + YX)F(1) \;=\; (\frac{1}{2}(2s)^2 + 0 - (2s)) \cdot F(1).$$

which is

$$\Omega f(g) \;=\; 2 \cdot (s^2 - s) \cdot f(g) \qquad \text{for } f \in I_s.$$

The relevance of this argument is that the Eisenstein series E_s generates an unramified principal series I_s under right translation. Since Casimir acts on I_s by $s(s-1)$, we see that E_s is an eigenfunction for the Laplacian Δ with eigenvalue $s(s-1)$.

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