RESEARCH STATEMENT

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Background

My research interest lies in applications of the spectral theory of automorphic forms to zeros of L-functions. A refined version of the spectral theory of automorphic forms plausibly has bearing on zeros of automorphic L-functions and other periods. My thesis proves that the zeros of a degree 4 L-function arise as parameters for the discrete spectrum of a self-adjoint perturbation $\widetilde{\Delta}_\theta$ of the Laplace-Beltrami operator.

The utility of the spectral theory of automorphic forms is powerfully illustrated by the following example. In 1977, Haas numerically computed eigenvalues λ of the invariant Laplacian

$$\Delta \ = \ y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$$

on $SL_2(\mathbb{Z})\backslash\mathfrak{H}$, parametrized as $\lambda_w = w(w-1)$. Haas listed the w-values, intending to solve the differential equation

$$(\Delta - \lambda_w)u = 0$$

Stark and Hejhal observed zeros of ζ and of an L-function on the list. This suggested an approach to proving the Riemann Hypothesis, since it seemed that zeros w of ζ might give eigenvalues $\lambda_w = w(w-1)$ of Δ. Since Δ is a self-adjoint, nonpositive operator, these eigenvalues would necessarily be nonpositive also, forcing either $\mathrm{Re}(w) = \frac{1}{2}$ or $w \in [0,1]$. Hejhal attempted to reproduce Haas' list with more careful computations, but the zeros failed to appear on Hejhal's list. Hejhal realized that Haas had solved the inhomogeneous equation

$$(\Delta - \lambda_w)u = \delta_\omega^{\text{afc}}$$

allowing a multiple of an automorphic Dirac δ on the right hand side. However, since solutions u_w of $(\Delta - \lambda)u = \delta_\omega^{\text{afc}}$ are not genuine eigenfunctions of the Laplacian, this no longer implied nonpositivity of the eigenvalues.

The natural question was whether the Laplacian could be perturbed so as to exhibit a fundamental solution as a legitimate eigenfunction for the perturbed operator. That is, one would want a variant $\Delta_?$ for which

$$(\Delta_? - \lambda_w)u_w \ = \ 0 \iff (\Delta - \lambda_w)u_w \ = \ C \cdot \delta_\omega^{\text{afc}}$$

Because of Colin de Verdiere's argument for meromorphic continuation of Eisenstein series as well as the iconic Lax-Phillips argument for discretization of the cuspidal spectrum [Lax-Phillips,1976, p.204-206], it was anticipated that $\Delta_? = \Delta^{\text{Fr}}$ would be a fruitful choice for a suitably chosen Friedrichs extension. Δ^{Fr} is self-adjoint, and therefore symmetric. This gave glimpses of a potential proof of the Riemann hypothesis.

Friedrichs extensions have the desired properties and they played a big part in another story, namely Colin de Verdiere's meromorphic continuation of Eisenstein series, though in that story, the distribution that appeared was the constant-term distribution. There, the spaces of interest were the orthogonal complements $L^2(\Gamma\backslash\mathfrak{H})_a$ to the spaces of pseudo-Eisenstein series whose test function is supported on $[0,\infty)$. Δ_a was Δ with domain $C_c^\infty(\Gamma\backslash\mathfrak{H})$ and constant term vanishing above height $y = a$. Δ^{Fr} was the Friedrichs extension of Δ_a to a self-adjoint unbounded operator on $L^2(\Gamma\backslash\mathfrak{H})_a$. In this way, a Friedrichs extension attached to the distribution on $\Gamma\backslash\mathfrak{H}$ given by

$$T_a(f) \;=\; (c_P f)(ia)$$

automatically places all eigenfunctions inside a $+1$-index Sobolev space. The Dirac δ on a two-dimensional manifold lies in a Sobolev space with index $-1-\epsilon$ for all $\epsilon > 0$, so by elliptic regularity, a fundamental solution lies in the $+1-\epsilon$-Sobolev space. This implies that a fundamental solution couldn't be an eigenfunction for any Friedrichs extension of a restriction of Δ described by boundary conditions.

Dissertation Work

This gives us a compelling reason to study the spectral theory of automorphic forms, as they encode simple yet elegant number-theoretic information. As rich as the SL_2 configuration is, it isn't indicative of the complexity of higher rank groups. Indeed, for SL_2 the residual spectrum of the Laplacian consists only of constants. For SL_4, there is a marked difference, in that Speh forms also enter into the discrete spectrum. This provides an incentive for setting up a finer harmonic analysis on higher rank groups; in particular one that does not involve gritty details. More to the point, it is anticipated that understanding spectral theory for higher rank groups will illuminate number-theoretic problems arising in lower-rank groups. This is part of the reason why the Haas-Hejhal episode sketched in the introduction struck a nerve in the 1970's and 1980's. There, $GL(2)$ spectral theory (Eisenstein series and cuspforms) seem to have bearing on $GL(1)$ (Riemann's zeta function).

The spectral theory for $G = SL_4(\mathbb{R})$ is considerably more complicated. Due to the plurality of parabolic subgroups, the continuous spectrum is spanned by many different kinds of pseudo-Eisenstein series. There is also residual spectrum, the so-called Speh forms. These arise as follows: Let ϕ be the function on $\mathbb{R}$ given by $\phi(t) = t^s$ and let f be a GL_2 cuspform with trivial central character. Let

$$\varphi(\begin{pmatrix} A & b \\ 0 & D \end{pmatrix}) = \phi(\Big|\frac{\det A}{\det D}\Big|^2)\cdot f(A)\cdot \overline{f}(D)$$

extending by right K-invariance to be made spherical. Define the $P^{2,2}$ Eisenstein series by

$$E_\varphi(g) \;=\; \sum_{\gamma\in P_k\backslash G_k} \varphi(\gamma g)$$

The theory of the constant term tells us that this Eisenstein series has a pole in the right half-plane $\text{Re}(s) > 1$, with a square-integrable residue, the Speh form Ψ_f.

My thesis is concerned with using Sobolev space techniques to gain traction on the zeros of a degree-4 L-function (under a plausible subconvexity bound). To this end, we consider the subspace V of $L^2(Z_{\mathbb{A}}G_k\backslash G_{\mathbb{A}})$ spanned by $2,2$ pseudo-Eisenstein series made with fixed cuspidal data f as well as the Speh form Ψ_f. Let V_a be the subspace of V consisting of those automorphic forms whose $2,2$ constant term vanishes above height $h = a$ and whose $1,3$ and $3,1$ constant term vanishes entirely. By completing automorphic test functions with respect to the s-th Sobolev norm

$$\langle f, f\rangle_{H^s} \;=\; \langle (1-\Delta)^s f, f\rangle_{L^2}$$

we get the s-th Sobolev space V^s. We will be concerned with V^{+1} as well as its Hilbert-space dual V^{-1}. Given a compactly-supported, Γ-invariant automorphic distribution $\tilde{\theta}$, whose projection θ to V is in V^{-1}, we let Δ_θ be Δ with domain kerθ, and let $\widetilde{\Delta}_\theta$ be the Friedrichs extension. My thesis shows that the discrete spectrum (if any) of the operator $\widetilde{\Delta}_\theta$ interlaces with the zeros of the constant term of the $2,2$ Eisenstein series. Such spacing is too regular to be compatible with pair correlation, so that the discrete spectrum of $\widetilde{\Delta}_\theta$ must be sparse or empty.

Future Work

The effect of considering n distributions is akin to that of considering n "boundary conditions". Manifest already in Sturm-Liouville problems, imposing boundary conditions has dramatic effects on the spectrum of an operator. As an example, consider the second-order differential equation

$$u^{''} - \lambda u \;=\; 0$$

on $(0, 2\pi)$ with boundary conditions $u(0) = u(2\pi) = 0$. This is equivalent to considering the differential equation in $L^2(\mathbb{R})$ given by

$$u^{''} - \lambda u \;=\; c\cdot(\delta_0 \pm \delta_{2\pi}) \quad \text{and} \quad (\delta_0 \pm \delta_{2\pi})u \;=\; 0$$

The solutions to this inhomogeneous equation are given by functions u with $u(x) = \sin(\frac{nx}{2})$ on $[0, 2\pi]$ and $u(x) = 0$ for $x \notin [0, 2\pi]$. This gives us discrete spectrum, in contrast to solving

$$u^{''} - \lambda u \;=\; 0$$

on $L^2(\mathbb{R})$ which has purely continuous spectrum.

Consider an n-dimensional subspace of H^{-1} whose intersection with H^0 is $\{0\}$. Take n linearly independent distributions $\theta_1, \theta_2, \ldots, \theta_n \in H^{-1}$ and solve

$$(\Delta - \lambda_w)u_i \;=\; \theta_i \quad \text{for } i = 1, 2\ldots, n$$

Let $\Delta_{\theta_1,\ldots,\theta_n}$ be Δ with domain $D = \bigcap \ker\, \theta_i$ and let $\widetilde{\Delta}_{\theta_1,\ldots,\theta_n}$ be the Friedrichs extension. Roughly as before, the discrete spectrum is nonempty if and only if

$$\det|\theta_i u_j| \;=\; 0$$

One immediate project is to study the dsicrete spectrum of $\widetilde{\Delta}_{\theta_1,\ldots,\theta_n}$ and investigate whether this operator is more willing to yield its mysteries than the operator in my thesis work.

Another project is to consider the period in [Jacquet-Lapid-Rogowski]. With $G = GL_4$ and $H \subset G$ a subgroup obtained as the fixed point set of an involution, the period is given by

$$\theta^H(\varphi) \;=\; \int_{H_k\backslash H_{\mathbb{A}}} \varphi(h)\, dh$$

This expression converges absolutely if φ is a cuspform. However, if φ is a cuspidal-data Eisenstein series, one needs to invoke a relative trace formula to show that the output gives an Euler product. I hope to investigate whether there is a spectral interpretation of the zeros of the resulting L-function, using Sobolev-space methods.

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