Syllabus for Complex Analysis: Math 8701-02; 2011-2012

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TEXT: *Ahlfors*, Complex Analysis (3rd ed).
POSSIBLE BUT UNLIKELY TEXT LATE 2ND SEMESTER: *Milnor*, Dynamics in One Complex Variable (3rd ed.), Princeton U. Press. (Good intro to Riemann surfaces)
SUGGESTED REFERENCE: *Gamelin*, Complex Analysis.
OTHER USEFUL REFERENCES:

- *Rudin*, Real and Complex Analysis (2nd ed)—a classic text combining treating real and complex analysis together;
- *Palka*, An Intro to Complex Function Theory (closely parallels Ahlfors but more expansive);
- *R. Remmert*, Classical Topics in Complex Function Theory;
- *R. Narasimhan and V. Nievergelt*, Complex Analysis in One Variable (from point of view of several complex variables).
- *E.B.Saff and A.D.Snider, Fundamentals of Complex Analysis with Appl.*
- *J.B.Conway, Functions of One Complex Variable* (simpler treatment).

All the above books are on reserve in the math lib.

Course Goal To provide a solid, classical foundation for the subject while exposing the many trails leading off in interesting directions.

1 Introduction

1. $\mathbb{C}$omplex numbers and their basic geometry
2. Stereographic projection: $\mathbb{C} \cup \{\infty\} \equiv \mathbb{S}^2$
3. Complex analysis from the point of view of advanced calculus
   - Complex derivatives $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$,
   - Conformal and quasiconformal mappings
   - Green's formula and applications

2 Analytic functions

1. Definitions and basic properties
2. Polynomial and rational functions

3. Power series; radius of convergence, Abel's theorem
4. Exponential and trig functions
5. Introduction to Riemann surfaces
6. Conformal mapping by elementary functions
7. Möbius transformations
   - classification
   - model examples, action in $\mathbb{C}$, quotient spaces
   - extension to 3D
   - cross ratio
   - symmetry
8. Introduction to hyperbolic geometry
9. Covering surfaces and the Riemann-Hurwitz formula

3 Core facts

3.1 A. Cauchy's Theorem and Integral Formula

1. Line integrals
2. Local Cauchy theorem; exceptional points
3. Convex Cauchy integral theorem
4. Winding number
5. General Cauchy theorem and Cauchy integral theorem using homology.

3.2 B. Local Properties

1. Removable singularities and their classification
2. Taylor's theorem with remainder
3. Zeros and poles; finite factorization
4. Essential singularities
5. The local mapping

3.3 C. Global Properties

1. Morara's theorem
2. Liouville's theorem
3. Maximum principle
4. Schwarz lemma
   - Schwarz-Pick lemma
5. Argument principle
6. Rouché's theorem; the local inverse

3.4 D. Residue Calculus

1. Residues
2. The residue theorem
3. Various explicit examples

3.5 E. Harmonic Functions

1. Definitions; rectangular and polar coordinate form of laplacian
2. Properties
3. du and the conjugate differential $^*du$
4. Mean values; maximum principle
5. Poisson integral formula and properties
6. Schwarz's formula
7. Sufficiency of mean value property; Harnak's inequality
8. Harnak's principle for increasing sequences

3.6 G. Reflection Principle

1. Reflections in lines and circles
2. Reflection principle

End of first semester, approximately.

4 Representations by series and products

4.1 Convergence

1. Pointwise, uniform convergence
2. Weierstrass convergence theorem
3. Hurwitz thm.

4.2 Power series

1. Taylor series
2. Laurent series
3. Analytic continuation

4.3 Other expansions

1. Partial fraction expansions (Mittag-Leffler thm.)
2. Infinite product expansions
    - Blaschke products; Jensen's formula

4.4 Two famous functions

1. Γ-function; Sterling's formula
2. ζ-function; Riemann hypothesis

5 Three famous theorems (without proofs)

1. Runge's approximation theorem
2. Mergelyan's approximation theorem
3. Fundamental theorem of holomorphic motion

6 Compact families

1. Normal families
2. Zalcman's criterion
    - Montel's theorem
    - Big Picard theorem

7 The Riemann mapping theorem

1. The proof
2. Boundary values
    - Prime ends
    - Extension across analytic boundary arcs
3. The Schwarz-Christoffel formula for polygonal maps
4. Rectangle mappings: intro to elliptic functions
5. Multiply connected regions and conformal moduli (statements only)
6. Discrete conformal mapping (statements only)
7. The uniformization theorem (statement and discussion only)

8 Dirichlet problem

1. Subharmonic functions
2. Statement of Dirichlet problem
    - The Perron family
    - Barriers
    - A general solution
3. Dirichlet's principle (statement only)
4. Domain functions: Green's function and harmonic measures

9 Elliptic curves; analysis on tori

9.1 A. Tori and the modular group

1. Lattices and period parallelograms
2. Tori
3. Conformal automorphisms of tori
4. The space of all tori
5. Uniformization of tori (statement only)
6. The Lattes example; dynamics on a torus (if time allows)
7. The modular group
   - Construction of the standard fundamental polygon
   - Action of the modular group on the space of tori
   - The congruence subgroup, mod 2
8. General Teichmüller spaces their modular groups (discussion only)

9.2 B. Functions on tori: elliptic functions

1. Definition and basic properties
2. The Weierstrass $\mathfrak{P}$-function
3. Elliptic curves

9.3 C. The modular function $\lambda(\tau)$

1. Definition and basic properties
2. The universal covering surface of the triply punctured sphere

10 Complex dynamics: Iteration of rational functions

It is unlikely we will get to this topic. That will be too bad as it takes off from the notion of normal family to study what happens if you iterate a rational function $R(z)$—what happens to the infinite sequence of points $R(z)$, $R^{(2)}(z) = R(R(z))$, $R^{(3)}(z), \ldots$ for each $z \in \mathbb{C}$. The simplest (but still highly nontrivial) case is $R(z) = z^2 + c$. Milnor's book is the definitive introduction.

10.1 A. Introduction and definitions

1. Newton's method
2. Critical points, classification of fixed points, periodic points
3. The Fatou and Julia sets

10.2 B. Properties of the Julia set

1. Repelling and parabolic fixed points
2. Connectedness, density of orbits

10.3 C. Classification of components of the Fatou set

1. No wandering domains (statement only)
2. attracting basins, rotation domains
3. Bottcher's theorem (proof if time allows)
4. Shishikura's theorem (statement only)

10.4 D. Quadratic polynomials

1. The special case of polynomials, quadratic polynomials
2. The structure of the Fatou sets
3. The Mandelbrot set (the parameter space)
    - Definition and properties
    - Proof that its complement is simply connected (time permitting)
    - Its interior structure; combinatorics
4. The big conjectures