AN ANALYTIC FUNCTION IN UNIT DISK WHOSE RADIAL CLUSTER SET AT EACH POINT ON THE UNIT CIRCLE IS THE WHOLE PLANE

0.1. **Definition.** The *radial cluster set* at a point $\zeta \in \partial\mathbb{D}$ for a function $f(z)$ analytic in $\mathbb{D}$ is defined as:

$$C_\zeta = \{w \in \mathbb{C} : w = \lim_{k\to\infty} f(r_k\zeta)\}, \text{ for sequences } \{r_k > 0,\ r_k \nearrow 1\}\}.$$

0.2. **A countable dense set in $\mathbb{C}$.** Choose a set Q which is both dense in $\mathbb{C}$ and such that each point of the set is repeated infinitely often in the sequence.

For example, set

$$Q_N = \{\frac{p}{q} + i\frac{r}{s} : |p|, |q|, |r|, |s| \le N\},$$

without zero denominators and the fractions being taken in lowest terms.

Then let Q_N^N be the set obtained by repeating each point of Q_N N-times. Note that

$$\cdots \subset Q_N^N \subset Q_{N+1}^{N+1} \subset \cdots .$$

Finally set $Q = \bigcup_{N=1}^\infty Q_N^N$. Let the sequence $\{w_m\}$ be an enumeration of Q.

0.3. **A construction.** Fix two sequences:

$$0 < s_1 < s_2 < \cdots \nearrow 1,$$

$$\pi > \epsilon_1 > \epsilon_2 > \cdots \searrow 0.$$

Construct the closed circular arc

$$\Gamma_k = \{s_k e^{i\theta}, \quad \epsilon_k \le \theta \le 2\pi\}.$$

- The sequence of closed arcs $\{\Gamma_k\}$ converges to the full circle $\partial\mathbb{D} = \{|z| = 1\}$.

0.4. **Induction argument.** Set

$$K_1 = \mathbb{C},$$

$$K_m = \Gamma_m \bigcup \{z : |z| < s_{m-1}\}.$$

Note that K_m is compact and its complement $\mathbb{S}^2 \setminus K_m$ is connected.

Correspondingly, apply Runge's theorem inductively to find polynomials $\{P_m\}$ with the properties:

$$P_1(z) = w_1,$$

$$|P_m(z) - P_{m-1}(z)| < \frac{1}{2^m}, \quad \text{for } |z| \le s_{m-1}, \ \text{AND}$$

$$|P_m(z) - w_m| < \frac{1}{2^m}, \quad \text{for } z \in \Gamma_m.$$

Since the disks are nested, it is automatically true that

$$|P_n(z) - P_{n-1}| < \frac{1}{2^n}, \quad \text{for } |z| \le s_{m-1}, \; n \ge m.$$

0.5. **Convergence.** When z is restricted to any of the disks $\{|z| < s_m\}$ we have

$$\sum_{m=1}^{\infty} |P_n(z) - P_{n-1}(z)| < \infty.$$

Therefore by the Weierstrass M-test,

$$f(z) := \lim P_n(z) = P_1(z) + \sum_{n=2}^{\infty} (P_n(z) - P_{n-1}(z))$$

where the series converges uniformly on each disk $\{|z| \le s_m\}$ to the limit function $f(z)$, which is therefore analytic in $\mathbb{D}$.

On the other hand, for *any* $z \in \Gamma_m$,

$$f(z) - w_m = \lim_{N\to\infty} \{(P_m(z) - w_m) + \sum_{n=m+1}^{N} (P_n(z) - P_{n-1}(z))\}.$$

Consequently

$$|f(z) - w_m| \le |P_m(z) - w_m| + \sum_{n=m+1}^{\infty} |P_n(z) - P_{n-1}(z)| < \frac{1}{2^m} + \sum_{n=m+1}^{\infty} \frac{1}{2^n} = \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^{m-1}}.$$

0.6. **Application to our problem.** For a given $\zeta \in \partial\mathbb{D}$, and a given point $w \in \mathbb{C}$ there exists a subsequence $\{w_{m_w}\} \subset \{w_m\} = Q$ and a corresponding sequence $\{s_{m_w} \nearrow 1\}$ such that

$$s_{m_w}\zeta \in \Gamma_m, \quad \lim_{m\to\infty} w_{m_w} = w.$$

To see why, consider the ray $\{t\zeta, \; 0 < t < 1\}$. For some N and all $k \ge N$, this ray intersects Γ_k at a point $s_k\zeta$. On the other hand since Q is dense, there is a subsequence $\{m_w\} \in Q$ with $w_{m_w} \to w$ as $m_w \to \infty$. As soon as $m_w \ge N$, set $k = m_w$.

We emphasize that different choices of ζ and w give rise to different subsequences of Q. The function $f(z)$ is unaffected as it was constructed from the enumeration of all Q.

0.7. **Conclusion.** Having chosen ζ and w we apply our formulas to the sequence of indices $\{m_w\}$ with $s_{m_w}\zeta \in \Gamma_m$ for all large m, and $\{w_{m_w}\} \subset Q$. We have in particular,

$$|f(s_{m_w}\zeta) - w_{m_w}| < \frac{1}{2^{m-1}}.$$

We also know that $\lim w_{m_w} = w$. The bottom line is that

$$\lim_{m\to\infty} f(s_{m_w}\zeta) = w.$$

Our construction, that follows Gamelin, p. 345, is complete. Have you ever seen such a weird analytic function? The family of analytic functions in $\mathbb{D}$ deserves our respect!