Homework 1

MATH 8660 Fall 2019

Due by 10/30/2019

Q1. Let X be an $n \times n$ matrix with i.i.d. entries with distribution having mean 0, variance 1, and compact support.

(a) Show that for each fixed integer $k \geq 1$, as $n \to \infty$,

$$\frac{1}{n^{k+1}}\mathbf{E}\Big[\mathrm{tr}(XX^T)^k\Big] \to C_k,$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number.

(b) Conclude that the expected ESD $\bar{L}_n := \mathbf{E}[L_n]$ of the matrix $n^{-1}XX^T$ converges in distribution to some probability measure ν. Identify ν.

Q2. (a) Using the bijection to the Dyck paths, prove the following recursion formula for the Catalan numbers:

$$C_k = \sum_{i=1}^{k} C_{i-1}C_{k-i} \quad \text{for } k \geq 1$$

with $C_0 = 1$.

(b) Using part (a), show that the Stieltjes transform G of the semicircle law satisfies the following equation:

$$G(z)^2 + zG(z) + 1 = 0 \quad \text{for any } z \in \mathbb{H}.$$

[Hint: $G(z) = -\sum_{k=0}^{\infty} \frac{C_k}{z^{1+2k}}$ for $|z| > 2$.]

Q3. Let $G \sim G(n, p = 1/2)$ be an Erdös-Rényi random graph and let A be the its adjacency matrix. Set $\bar{A} = A - \mathbf{E}[A]$. The goal of this exercise to show that $\|\bar{A}\| := \sup_{x:\|x\|_2=1} |x^T\bar{A}x| = O(\sqrt{n})$ with high probability without resorting to the improved moment bound result.

(a) Use Azuma-Hoeffding inequality to show that for a fixed unit vector $x \in \mathbb{R}^n$,

$$\mathbf{P}\big(|x^T\bar{A}x| > t\big) \leq 2e^{-t^2} \quad \text{for any } t > 0.$$

(b) Let B be a symmetric matrix of size n and let v be a unit eigenvector of B whose eigenvalue has absolute value $\|B\|$. Then for any unit vector x such that $x^Tv \geq \sqrt{3}/2$, we have

$$|x^TBx| \geq \frac{1}{2}\|B\|.$$

(c) Let X be chosen uniformly from S^{n-1}. Then for any fixed unit vector v,

$$\mathbf{P}\big(X^Tv \geq \sqrt{3}/2\big) \geq \frac{1}{\sqrt{\pi}n2^{n-1}}.$$

(d) Let X be chosen uniformly from S^{n-1}, independent of $\bar{A}$. Fix $t > 0$. Use parts (b) and (c) to derive that

$$\mathbf{P}_{\bar{A},X}\Big(|X^T\bar{A}X| \geq \frac{1}{2}\|\bar{A}\|, \;\; \|\bar{A}\| \geq t\Big) \geq \frac{1}{\sqrt{\pi}n2^{n-1}}\mathbf{P}_{\bar{A}}\big(\|\bar{A}\| \geq t\big).$$

On the other hand, use part (a) to show that

$$\mathbf{P}_{\bar{A},x}\Big(|x^T\bar{A}x| \geq \frac{1}{2}\|\bar{A}\|, \;\; \|\bar{A}\| \geq t\Big) \leq 2e^{-t^2/4}.$$

(e) Deduce that for any $t > 0$,

$$\mathbf{P}_{\bar{A}}\big(\|\bar{A}\| \geq t\big) \leq \sqrt{\pi} n 2^n e^{-t^2/4},$$

and hence conclude that $\|\bar{A}\| = O(\sqrt{n})$ except for an exponentially small probability.

Q4. The goal of this exercise is to provide an alternative way of showing the key step in the Stieltjes transform proof of Wigner's semicircle law for *Gaussian Orthogonal Ensemble*. The proof relies crucially on the Gaussian integration by parts formula.

Let $X = ((X_{ij}))$ be an $n \times n$ GOE matrix, i.e., $X_{ij}, i < j$ are i.i.d. $N(0,1)$ and X_{ii} are i.i.d. $N(0,2)$, independent of $X_{ij}, i < j$. Let L_n be the ESD of $Y := n^{-1/2} X$ and let $G_{L_n}(z) := \int (x-z)^{-1} dL_n(x), z \in \mathbb{H}$ be the Stieltjes transform of L_n. Throughout the rest of the exercise, we fix $z \in \mathbb{H}$.

(a) Prove the resolvent identity: for any $n \times n$ symmetric matrices A and B,

$$(A - zI)^{-1} - (B - zI)^{-1} = (B - zI)^{-1}(B - A)(A - zI)^{-1}.$$

(b) Using the resolvent identity show that
(i) $(Y - zI)^{-1} = -\frac{1}{z} + \frac{1}{z} Y (Y - zI)^{-1}$.
(ii) $\frac{\partial (Y - zI)^{-1}}{\partial Y_{ij}} = -(Y - zI)^{-1} \Delta_{ij} (Y - zI)^{-1}$, where $\Delta_{i,j} := e_i e_j^T + e_j e_i^T$ if $i \neq j$ and $\Delta_{i,i} := e_i e_i^T$ and $e_1, e_2, \ldots, e_n$ are the standard coordinate vectors of $\mathbb{R}^n$.

(c) Using (b)(i) show that

$$\frac{1}{n}\mathbf{E}[\mathrm{tr}(Y - zI)^{-1}] = -\frac{1}{z} + \frac{1}{nz}\mathbf{E}[\mathrm{tr}(Y(Y - zI)^{-1})] = -\frac{1}{z} + \frac{1}{nz}\sum_{i,j}\mathbf{E}[Y_{ij}(Y - zI)^{-1}_{ji}].$$

(d) Prove the Stein's identity or Gaussian integration by parts formula.
Let $Z \sim N(0, \sigma^2)$. Then for any differentiable function f which grows no faster than a polynomial near $\pm\infty$, we have

$$\mathbf{E}[Zf(Z)] = \sigma^2 \mathbf{E}[f'(Z)].$$

(e) Use Stein's identity to show that

$$\sum_{i,j}\mathbf{E}[Y_{ij}(Y - zI)^{-1}_{ji}] = -\frac{1}{n}\mathbf{E}\Big(\sum_i 2((Y - zI)^{-1}_{ii})^2 + \sum_{i \neq j}\big((Y - zI)^{-1}_{ji}(Y - zI)^{-1}_{ij} + (Y - zI)^{-1}_{jj}(Y - zI)^{-1}_{ii}\big)\Big).$$

(f) Show that

$$\frac{1}{n}\mathbf{E}[\mathrm{tr}(Y - zI)^{-1}] = -\frac{1}{z} - \frac{1}{n^2 z}\mathbf{E}[\mathrm{tr}(Y - zI)^{-2}] - \frac{1}{z}\mathbf{E}[(\tfrac{1}{n}\mathrm{tr}(Y - zI)^{-1})^2].$$

(g) Argue that $|\frac{1}{n}\mathrm{tr}(Y - zI)^{-2}| \leq \mathrm{Im}(z)^{-2}$ and conclude, assuming that $\mathrm{Var}(G_{L_n}(z)) \to 0$ which we had proved in the lecture using Azuma-Hoeffding, that as $n \to \infty$

$$(\mathbf{E}[G_{L_n}(z)])^2 + z\mathbf{E}[G_{L_n}(z)] + 1 \to 0.$$

Q5. Consider a $n \times n$ Jacobi matrix T_n whose all diagonal entries $a_k = 0$ and off-diagonal entries $b_k = 1$. In other words, T_n is the adjacency matrix of a path graph with n vertices. Let μ_n be the empirical spectral distribution of T_n.

(a) Show that for each fixed integer $k \geq 1$, as $n \to \infty$,

$$\int x^k d\mu_n \to \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \binom{k}{k/2} & \text{if } k \text{ is even.} \end{cases}$$

(b) Show that $\mu_n \xrightarrow{d} \mu$, where μ is the arc-sine law on $[-2, 2]$ with density $\rho(x) = \frac{1}{\pi\sqrt{4 - x^2}}, |x| \leq 2$.

(c) If ν_n is the spectral measure of T_n at e_1, then compute the distributional limit of ν_n. What about the spectral measure of T_n at $e_{\lfloor n/2 \rfloor}$?