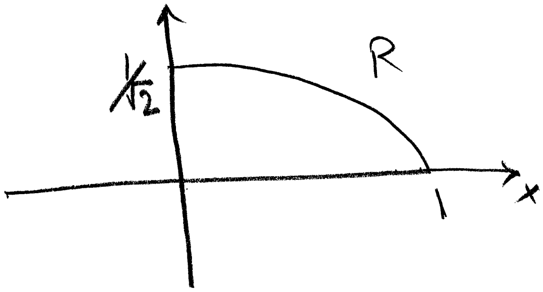
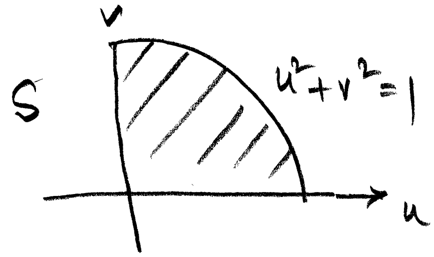
Solution to Practice Problems for Midterm 3.



Change of variables :-

$$u = x \qquad x = u$$
$$v = \sqrt{2}\, y \qquad y = \frac{1}{\sqrt{2}} v$$



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{2}}$$

By change of variable formula,

$$\iint_R \sin(x^2+2y^2)\, dx\, dy = \iint_S \sin(u^2+v^2) \left|\frac{1}{\sqrt{2}}\right| \cdot du\, dv$$

$$= \frac{1}{\sqrt{2}} \iint_S \sin(u^2+v^2)\, du\, dv$$

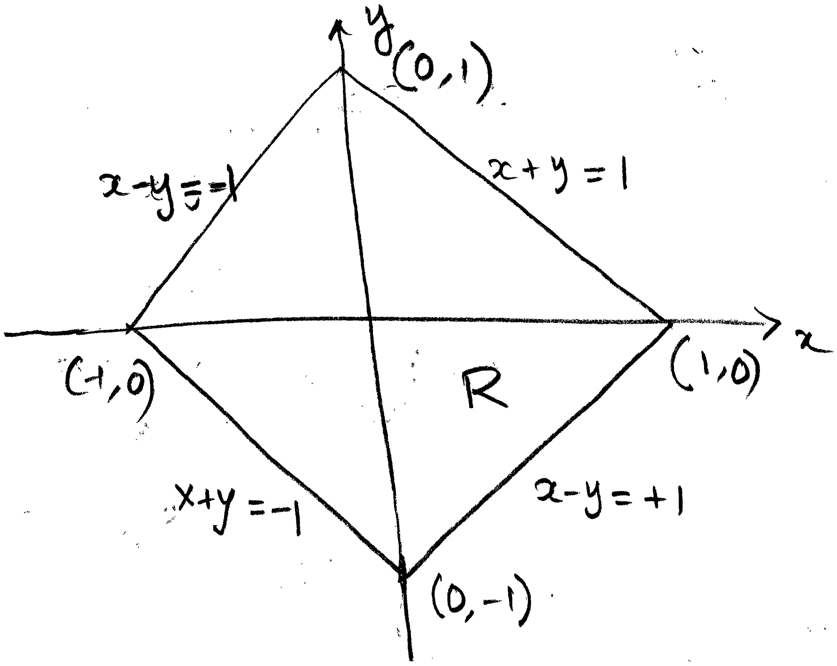
$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \int_0^1 \sin(r^2)\, r . dr\, d\theta \qquad u = r\cos\theta, \quad v = r\sin\theta$$

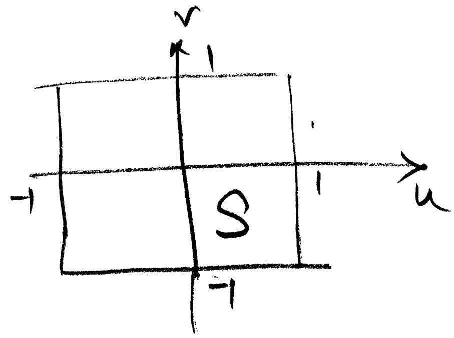
$$= \frac{1}{\sqrt{2}} \times \int_0^{\pi/2} d\theta \times \int_0^1 \sin(r^2)\, r . dr$$

$$= \frac{1}{\sqrt{2}} \cdot \pi/2 \cdot \int_0^1 \sin(t)\, \frac{dt}{2} \qquad r^2 = t$$

$$= \frac{1}{\sqrt{2}} \cdot \pi/2 \cdot \frac{1}{2} \cdot -\cos t \Big|_0^1 \qquad 2r\,dr = dt$$

$$= \frac{\pi}{4\sqrt{2}} (1 - \cos(1)).$$





Set $u = x+y$, $v = x-y$, then $\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/2, \quad dA = \frac{1}{2}\,du\,dv$$

$$\iint_R \left(\frac{x-y}{x+y+2}\right)^2 dx\,dy = \iint_S \frac{v^2}{(u+2)^2}\,\frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\int_{v=-1}^{1}\int_{u=-1}^{1} \frac{v^2}{(u+2)^2}\,du\,dv$$

$$= \frac{1}{2}\int_{v=-1}^{1} v^2 \cdot \left.\frac{(u+2)^{-1}}{-1}\right|_{u=-1}^{u=1} dv$$

$$= \frac{1}{2}\int_{v=-1}^{1} v^2 \cdot \left(1 - \frac{1}{3}\right) dv = \frac{1}{3}\int_{v=-1}^{1} v^2\,dv$$

$$= \frac{2}{9}.$$

To show

3. (i) $\int_C 2x \sin(y)\,dx + (x^2 \cos y - 3y^2)\,dy$

is indep of path, we need to show that

$\vec{F} = \langle \underbrace{2x \sin y}_{P}, \underbrace{x^2 \cos y - 3y^2}_{Q} \rangle$ is Conservative.

$\frac{\partial Q}{\partial x} = 2x \cos y$, $\frac{\partial P}{\partial y} = 2x \cos y$

hence $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$. $\vec{F}$ is defined on $\underbrace{\mathbb{R}^2}_{\text{simply-connected}}$.

Hence, $\vec{F}$ is Conservative.

(ii) let's find a potential f for $\vec{F}$.

$f_x = 2x \sin y$ — ① $f_y = x^2 \cos y - 3y^2$ — ②

Integrate ① w.r.t. x,

$f(x,y) = x^2 \sin y + g(y)$ — ③

Thus $f_y = x^2 \cos y + g'(y)$ — ④

By ② & ④ $g'(y) = -3y^2$, $g(y) = -y^3 + C$.

Thus, $f(x,y) = x^2 \sin y - y^3 + C$.

when C is any path from $(1,0)$ to $(-1, \pi/2)$

the by fundamental theorem of line integral,

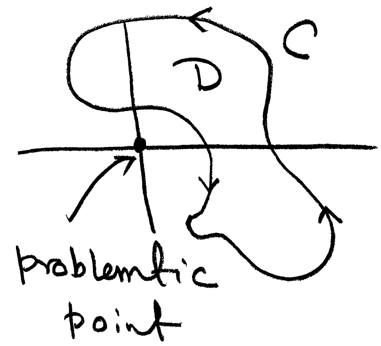
$\int_C \vec{F} \cdot d\vec{r} = f(-1, \pi/2) - f(1,0)$

$= (1 \cdot 1 - (\pi/2)^3 + C) - (0 - 0 + C)$

$= 1 - \pi^3/8$.

4. $\vec{F} = \left\langle \underbrace{\frac{-y}{x^2+y^2}}_{P}, \underbrace{\frac{x}{x^2+y^2}}_{Q} \right\rangle$

P, Q have cont partial derivatives everywhere in $\mathbb{R}^2$ except $(0,0)$.



(a). By Green's thm,

$$\oint_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \iint_D \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right] dA$$

$$= \iint_D \left[\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2}\right] dA = \iint_D 0\, dA = 0.$$

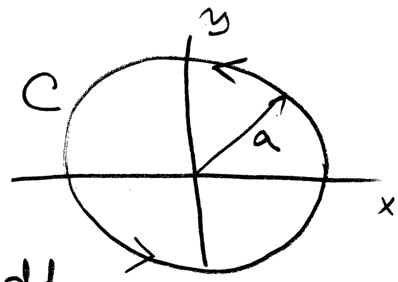
$$\left[\frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{1.(x^2+y^2) - x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},\right.$$

$$\left.\frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = \frac{-1(x^2+y^2) - 2y(-y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}\right].$$

(b). Note that Green's thm can not be applied here since $\underbrace{\text{the curve}}_{\text{the circle}}$ contains the point $(0,0)$ inside. Instead, we will compute the line integral directly.

$C: \vec{r}(t) = \langle a\cos t, a\sin t \rangle,$

$t \in [0, 2\pi]$



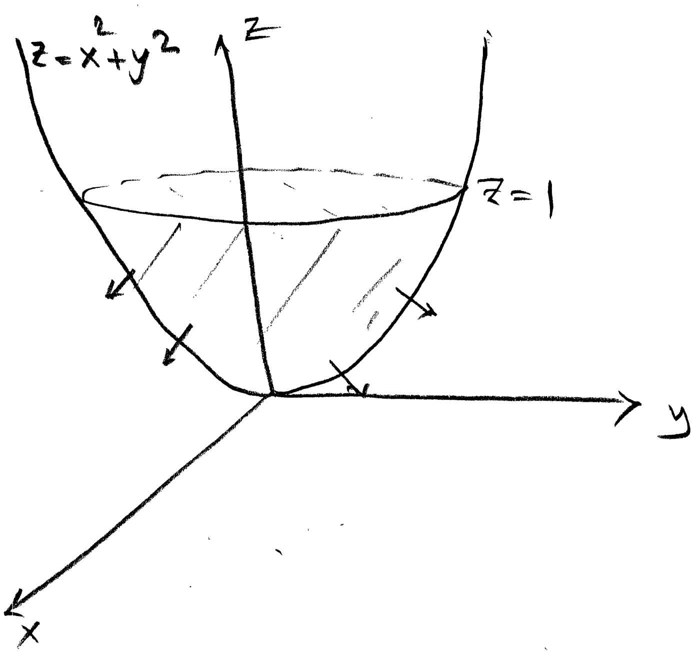
$dx = -a\sin t\, dt, \quad dy = a\cos t\, dt$

$$\oint_C \frac{-y}{x^2+y^2} \cdot dx + \frac{x}{x^2+y^2}\, dy$$

$$= \int_0^{2\pi} \frac{-a\sin t\,(-a\sin t) + a\cos t \cdot a\cos t}{a^2}\, dt$$

$$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = \int_0^{2\pi} 1 \cdot dt = 2\pi.$$

5.



$S: \vec{r}(x,y) = \langle x, y, \underbrace{x^2+y^2}_{g(x,y)} \rangle, \quad x^2+y^2 \le 1$

$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle = \langle -2x, -2y, 1 \rangle$ upward orientation

6

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{x^2+y^2\le 1} \langle 2x, -2y, 1\rangle \cdot \langle 2x, 2y, -1\rangle \, dx\,dy$$

downward orientation

$$= \iint_{x^2+y^2\le 1} \left(4(x^2-y^2) - 1\right) dx\,dy$$

$$= \int_0^1 \left[\int_0^{2\pi} 4r^2(\cos^2\theta - \sin^2\theta)\right) - 1 \cdot r \cdot dr\,d\theta$$

$$= 4\int_0^1 \int_0^{2\pi} \cos 2\theta \cdot d\theta \cdot r^3 \cdot dr - \pi$$

$$= 4 \times \frac{\sin 2\theta}{2} \Big|_0^{2\pi} - \pi$$

$$= \frac{4}{2}(\sin 4\pi - \sin 0) - \pi = -\pi.$$

6. Let $\nabla f = \vec{F}$.

$$f_x = e^x \cos y - \ln z \quad \text{— (1)}$$

Integrate w.r.t. x

$$f(x,y,z) = e^x \cos y - x \ln z + g(y,z)$$

$$f_y = -e^x \sin y + g_y(y,z) \quad \text{— (2)}$$

Also $f_y = -e^x \sin y + 2yz$ — (3)

(2) & (3) $\Rightarrow$ $g_y = 2yz$

Integrate w.r.t. y

$$g(y,z) = y^2 z + h(z).$$

Thus, $f(x,y,z) = e^x \cos y - x \ln z + y^2 z + h(z)$

$$f_z = -x/z + y^2 + h'(z) \quad \text{— (4)}$$

Also, $f_z = -x/z + y^2$

$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C.$

Thus $f(x,y,z) = e^x \cos y - x \ln z + y^2 z + C.$

7. (i) $\vec{F} = \langle \underbrace{e^y + y^2 e^x}_{P}, \underbrace{x e^y + 2y e^x}_{Q} \rangle$

$$\frac{\partial Q}{\partial x} = e^y + 2y e^x, \quad \frac{\partial P}{\partial y} = e^y + 2y e^x$$

Hence $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$, and $\vec{F}$ is defined on the entire $\mathbb{R}^2$ which is simply-connected.

Hence, $\vec{F}$ is conservative.

(ii).
$$f_x = e^y + y^2 e^x \quad \text{— (1)}$$
$$f_y = x e^y + 2y e^x \quad \text{— (2)}$$

$$f(x,y) = x e^y + y^2 e^x + g(y) \quad \text{— (3)}$$

or, $f_y = x e^y + 2y e^x + g'(y)$ — (4)

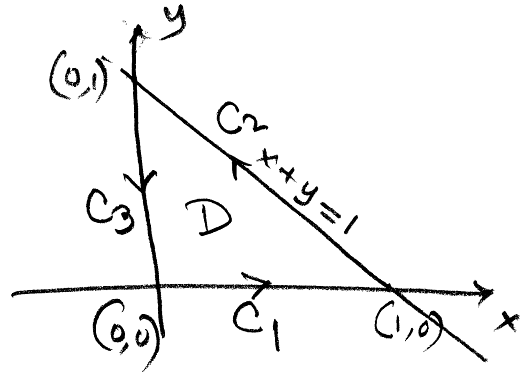
Compare (3) & (4) to obtain,

$$g'(y) = 0$$
$$\Rightarrow \quad g(y) = C.$$

Hence, $f(x,y) = x e^y + y^2 e^x + C$.

8. (a)

$$\oint_C x^2 dx + xy\,dy$$



$$= \int_{C_1} x^2 dx + xy\,dy + \int_{C_2} x^2 dx + xy\,dy$$

$$+ \int_{C_3} x^2 dx + xy\,dy$$

C_1 : parametrize : $y = 0$, $x = x$, $x \in [0,1]$

$dy = 0.dx \quad dx = 1.dx$,

$$\int_{C_1} x^2 dx + xy\,dy = \int_0^1 x^2 . dx + 0 . dx = \int_0^1 x^2 dx = \tfrac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

C_2 : parametrize : $\vec{r}(t) = t\langle 0,1\rangle + (1-t)\langle 1,0\rangle = \langle 1-t, t\rangle$, $t \in [0,1]$

$$\int_{C_2} x^2 dx + xy\,dy = \int_0^1 (1-t)^2 - dt + t(1-t).dt$$

$$= \int_0^1 \left[(t - t^2) - (1 - 2t + t^2)\right] dt$$

$$= \int_0^1 (-1 + 3t - 2t^2)\,dt = -t + \frac{3t^2}{2} - \frac{2}{3}t^3\Big|_0^1$$

$$= -1 + 3/2 - 2/3$$

$$= \frac{-6+9-4}{6} = -\frac{1}{6}.$$

C_3 : parametrize , $x = 0$, $y = -t$, $t \in [0,1]$

$dx = 0.dt$, $dy = -dt$

$$\int_{C_3} x^2 dx + xy\,dy = \int_0^1 0^2 dt + 0.dt = 0.$$

$$\int_{C_1} + \int_{C_2} + \int_{C_3} = \frac{1}{3} - \frac{1}{6} + 0 = \frac{1}{6}.$$

(b). Green's thm

$$\oint_C x^2 dx + xy\,dy = \iint_D \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\, dA$$

$$= \iint_D y . dA$$

$$= \int_0^1 \int_0^{1-y} y . dx\,dy = \int_0^1 y - y^2\, dy$$

$$= \frac{y^2}{2} - \frac{y^3}{3} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

9. $\vec{F} = \langle e^{-x}\sin y,\ xe^{-z}\sin y,\ xe^{-z}\cos y \rangle$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(-xe^{-z}\sin y + xe^{-z}\sin y\right)\vec{i} -$$

$$\left(e^{-z}\cos y \quad -0\right)\vec{j} +$$

$$\left(e^{-z}\sin y - e^{-x}\cos y\right)$$

$$= \langle 0, -e^{-z}\cos y,\ e^{-z}\sin y - e^{-x}\cos y \rangle$$

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= -e^{-x}\sin y + xe^{-z}\cos y - xe^{-z}\cos y$$

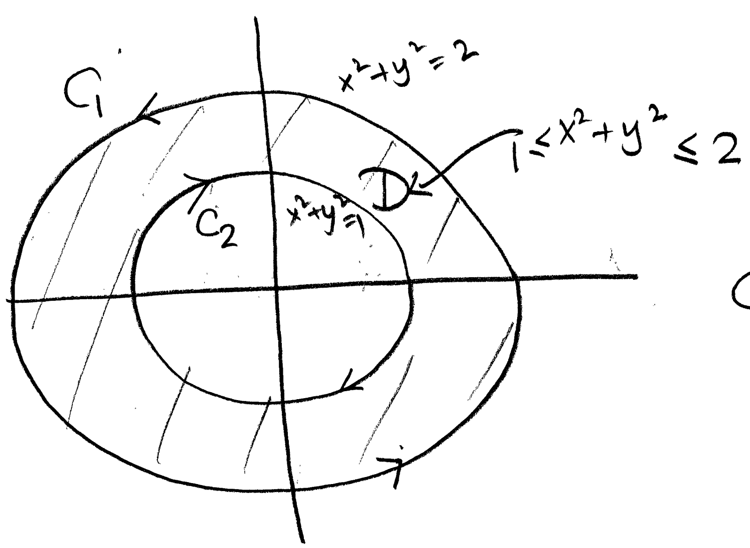
$$= -e^{-x}\sin y.$$

Since $\nabla \times \vec{F} \neq 0$, and $\nabla \times (\nabla f) = 0$ always

$F \neq \nabla f$. Hence $\vec{F}$ is not conservative.

Since. $\nabla \cdot \vec{F} = 0$ and $\nabla \cdot (\nabla \times \vec{G}) = 0$ always

$\vec{F} \neq \text{Curl}(\vec{G})$ for any $\vec{G}$.



$C = C_1 \cup C_2$.

Green's thm

$$\oint_C y^3\,dx - x^3\,dy = \iint_D \frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\,dA$$

$$= -3 \iint_D (x^2+y^2)\,dA$$

(12)

$$= -3 \int_{0}^{2\pi} \int_{r=1}^{\sqrt{2}} r^2 . r . dr\, d\theta$$

$$= -3 \int_{0}^{2\pi} d\theta \times \int_{1}^{\sqrt{2}} r^3 . dr$$

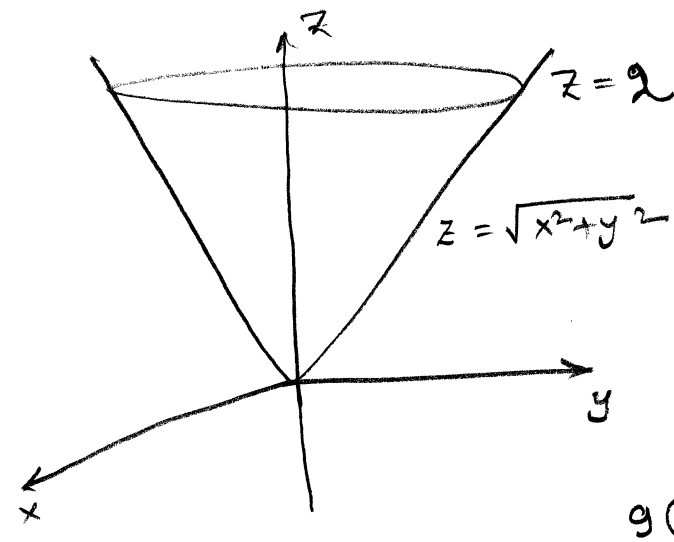
$$= -3 \times 2\pi \times \left. \frac{r^4}{4} \right|_1^{\sqrt{2}}$$

$$= -3 \times 2\pi \times \frac{1}{4}(4-1)$$

$$= -\frac{9\pi}{2}$$

Note –
Here we apply Green's thm for region with a hole, which is justified (see the discussion in page 1088).

11.



S: parametrize $\vec{r}(x,y) = \langle x, y, \underbrace{\sqrt{x^2+y^2}}_{g(x,y)} \rangle$,

$x^2+y^2 \leq 4$.

$$\text{Area}(S) = \iint_{x^2+y^2 \leq 4} \sqrt{1 + g_x^2 + g_y^2}\, . dx\, dy$$

$$g_x = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}$$

$$g_y = \frac{y}{\sqrt{x^2+y^2}}$$

$$1 + g_x^2 + g_y^2 = 1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}$$

$$= 2.$$

$$\text{Area}(S) = \iint_{x^2+y^2 \le 4} \sqrt{2}\,.\,dx\,dy$$

$= \sqrt{2} \times$ area of circle of radius 2

$$= \sqrt{2} \times \pi . 2^2 = 4\sqrt{2}\,\pi$$