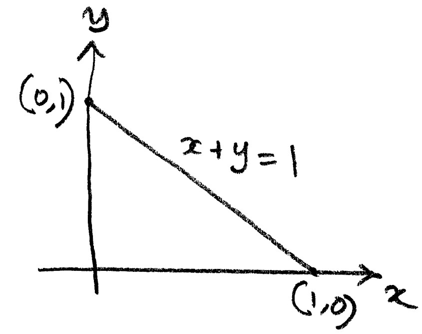
1. (20 points) Evaluate the following double integral

$$\iint_R e^{\frac{x-y}{x+y}}\, dA,$$

where R is the triangle with vertices $(0,0), (1,0)$ and $(0,1)$.

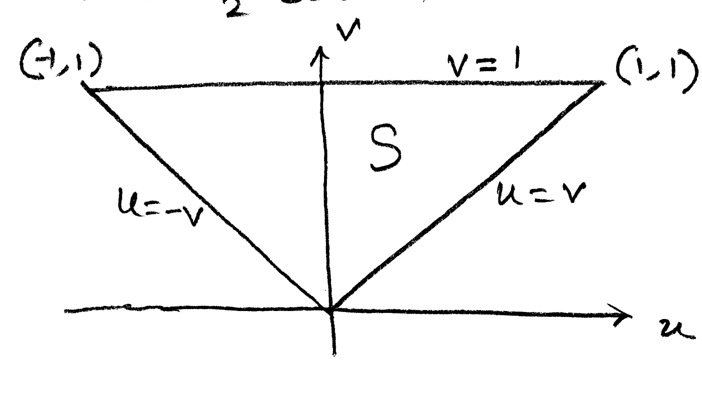


Set $x - y = u, \quad x + y = v$

Solving, $x = \frac{u+v}{2}, \quad y = \frac{1}{2}(v-u).$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Thus $dA = |J(u,v)|\, du dv = \frac{1}{2} du dv$



$$\iint_R e^{\frac{x-y}{x+y}}\, dA =$$

$$\iint_S e^{u/v} \frac{1}{2} du dv$$

$$= \int_{v=0}^{1} \int_{-v}^{v} e^{u/v} \frac{1}{2} du dv$$

$$= \int_{v=0}^{1} \frac{v}{2} e^{u/v} \Big|_{u=-v}^{u=v} \cdot dv = \int_0^1 \frac{v}{2}(e - e^{-1})$$

$$= \frac{v^2}{4}(e - e^{-1}) \Big|_0^1$$

$$= \frac{1}{4}\left(e - \frac{1}{e}\right).$$

2. (15 points) Show that line integral given by

$$\oint_C xy^2dx + (x^2y + 3x)dy$$

around *any* circle C (in counterclockwise orientation) depends only on the area of the circle and not on its location in the plane.

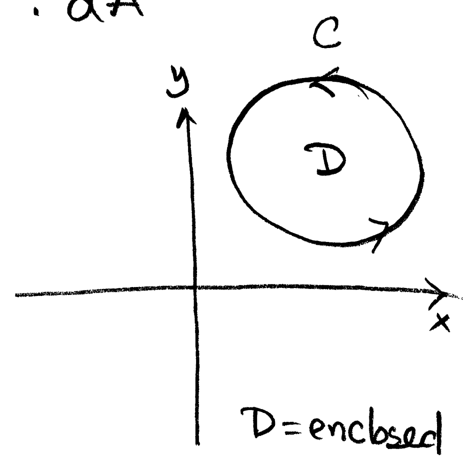
By Green's thm, $\oint_C xy^2 dx + (x^2y+3x)dy$

$$= \iint_D \frac{\partial}{\partial x}(x^2y+3x) - \frac{\partial}{\partial y}(xy^2) \cdot dA$$

$$= \iint_D (2xy+3 - 2xy) \, dA$$

$$= \iint_D 3 \cdot dA = 3 \iint 1 \cdot dA$$

$$= 3 \text{ area}(D).$$



D = enclosed disk

only depends on the area of the circle, not its center.

3. (15 points) Find a potential function for the vector field $\vec{F} = \langle 3x^2y + y^2, x^3 + 2xy + 3y^2 \rangle$.

Let $\vec{F} = \nabla f$, i.e. $f_x = 3x^2y + y^2$ — ①

$f_y = x^3 + 2xy + 3y^2$ — ②

Integrate ① w.r.t. x,

$$f(x,y) = \int 3x^2y + y^2 \, dx$$

$$= x^3y + xy^2 + g(y) \quad \text{— ③}$$

Hence, $f_y = x^3 + 2xy + g'(y)$ — ④

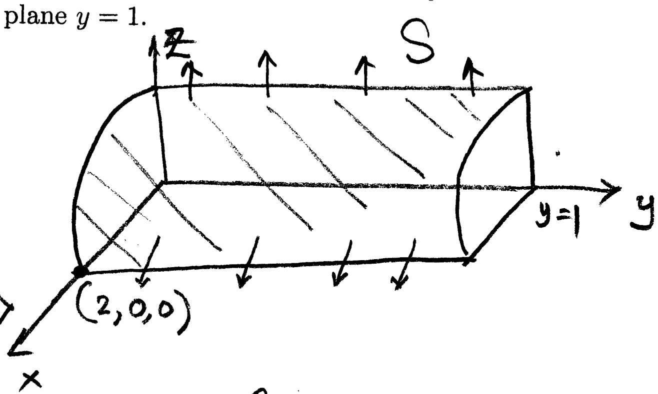
Comparing ③ & ④ $\Rightarrow$

$$g'(y) = 3y^2$$

Hence $g(y) = \int 3y^2 \, dy = y^3 + C$

Hence, $f(x,y) = x^3y + xy^2 + y^3 + C$.

4. (20 points) Find the flux of $\vec{F} = y\vec{i} + x\vec{j} + z\vec{k}$ outward through the portion of the cylinder $x^2 + z^2 = 4$ in the first octant and bounded by the plane $y = 1$.



parametrize S :

$\vec{r}(\theta, y) = \langle 2\cos\theta, y, 2\sin\theta \rangle$

$0 \leq y \leq 1$, $0 \leq \theta \leq \pi/2$ (since $x \geq 0, z \geq 0$)

$$\vec{r}_\theta \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2\sin\theta & 0 & 2\cos\theta \\ 0 & 1 & 0 \end{vmatrix} = \langle -2\cos\theta, 0, -2\sin\theta \rangle$$

inward orientation

[since at say $\theta = 0$, $y = 0$ i.e. $(2,0,0)$

$\vec{r}_\theta \times \vec{r}_y = \langle -2, 0, 0 \rangle$, points towards origin]

Flux =

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi/2}\int_0^1 \langle y, 2\cos\theta, 2\sin\theta \rangle \langle 2\cos\theta, 0, 2\sin\theta \rangle \, dy\, d\theta$$

$$= \int_0^{\pi/2}\int_0^1 2y\cos\theta + 4\sin^2\theta \, dy\, d\theta$$

$$= \int_0^{\pi/2} 2\cos\theta \cdot \frac{y^2}{2}\Big|_0^1 + 2(1 - \cos 2\theta) \cdot y\Big|_0^1 \, . d\theta$$

$$= \int_0^{\pi/2} 2 + \cos\theta - 2\cos 2\theta \, . d\theta = 2\theta + \sin\theta - \sin 2\theta \Big|_{\theta=0}^{\theta=\pi/2}$$

$$= \pi + 1$$

5. (15 points) Find the equation of the tangent plane to the surface $\vec{r}(u,v) = \langle u^2 - v^2, v^3, 2uv\rangle$ at the point $P = (0, -1, -2)$.

let's first find u, v corresponding to P,

$u^2 - v^2 = 0$, —① $\quad v^3 = -1$, —② $\quad 2uv = -2$. —③

From ②, $v = -1$, From ③ $2 \cdot u \cdot (-1) = -2$

or, $u = 1$

$\vec{r}_u = \langle 2u, 0, 2v \rangle$, $\vec{r}_v = \langle -2v, 3v^2, 2u \rangle$

$$\vec{r}_u(1,-1) \times \vec{r}_v(1,-1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -2 \\ +2 & 3 & 2 \end{vmatrix} = \langle 6, -8, 6 \rangle$$

The tangent plane passes through $(0,-1,-2)$ and has a normal vector $\langle 6, -8, 6 \rangle$

Its equation :- $6(x-0) - 8(y-(-1)) + 6(z-(-2)) = 0$

or, $6x - 8(y+1) + 6(z+2) = 0$

or, $6x - 8y + 6z = -4$

or, $3x - 4y + 3z = -2$

6. (15 points) Find the work done by the force field

$$\vec{F}(x,y) = \langle ye^{xy}, xe^{xy} \rangle$$

as it acts on a particle moving from $P = (-1,0)$ to $Q = (1,0)$ along the semicircular arc C given by $\vec{r}(t) = \langle -\cos t, \sin t \rangle$, $0 \le t \le \pi$.

Work done $= \int_C \vec{F} \cdot d\vec{r}$ where C is the given curve. This line integral is difficult to compute directly. But $\vec{F}$ is conservative!

(since $\frac{\partial}{\partial x}(xe^{xy}) - \frac{\partial}{\partial y}(ye^{xy}) = (e^{xy} + xye^{xy}) - (e^{xy} + xye^{xy}) = 0$]

Let's find a potential f s.t. $\nabla f = \vec{F}$.

$f_x = ye^{xy}$, or, $f(x,y) = e^{xy} + g(y)$.

$\Rightarrow f_y = xe^{xy} + g'(y)$

On the other hand, $f_y = xe^{xy}$. Hence $g'(y) = 0$
or, $g(y) = C$.

Thus $f(x,y) = e^{xy} + C$.

By the fundamental theorem of line integral,

$$\int_C \vec{F} \cdot d\vec{r} = f(1,0) - f(-1,0)$$

$$= (e^{1 \cdot 0} + C) - (e^{-1 \cdot 0} + C)$$

$$= 1 - 1 = 0$$