Gröbner Geometry of Schubert and Grothendieck transition formulae

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Gröbner bases and Gröbner geometry

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring and $I \subseteq R$ an ideal.

Hilbert's basis theorem: $I = \langle f_1, \ldots, f_k \rangle$.

Pick a term order $\prec$ on the monomials in R. This determines an **initial term** $\text{in}(f)$ for every $f \in I$.

Let

$$\text{in}(I) = \langle \text{in}(f) : f \in I \rangle.$$

Moreover, $\{f_1, \ldots, f_k\}$ is a **Gröbner basis** of I with respect to $\prec$ if

$$\text{in}(I) = \langle \text{in}(f_1), \ldots, \text{in}(f_k) \rangle$$

Gröbner degenerations

$X = \mathsf{Spec}(R/I)$ is the affine scheme of I.

$\mathsf{in}(X) = \mathsf{Spec}(R/\mathsf{in}(I))$ is the **initial scheme** of X

Algebraically, we use Gröbner bases to deduce properties of R/I from the simpler $R/\mathsf{in}(I)$

Geometrically, a Gröbner basis corresponds to a flat deformation of X under the action of $\mathbb{C}^*$ until it degenerates (breaks) into a union of coordinate subspaces.

Some invariants of X such as the Hilbert series are necessarily preserved under such degenerations.

E.g., properties such as reducedness, equidimensionality, Cohen-Macaulayness, will follow for X **if** one can degenerate X to some $\mathsf{in}(X)$ with these properties.

One step in a Gröbner degeneration

Theme: Gröbner degeneration in stages.

Let X be a closed subscheme of a vector space $H \oplus L$ (for **hyperplane** and **line**).

Let $\mathbb{C}^*$ act on X by rescaling the second component:

$$t \cdot (\overrightarrow{x}, y) := (\overrightarrow{x}, ty)$$

The flat limit

$$X' := \lim_{t \to 0} t \cdot X$$

is a "slow-motion" projection of X onto H

X' obviously contains the actual projection of X onto H.

But it actually contains more.

Geometric vertex decompositions

Consider the closure $\overline{X}$ inside $H \times (L \cup \{\infty\})$. Define

$$\Lambda := \overline{X} \cap (H \times \{\infty\}) \subseteq H \times \{\infty\} \cong H.$$

Our results are in terms of the following framework:

Theorem 1 (Knutson-Miller-Y.) *The support of the limit scheme X' is exactly*

$$(\Pi \times \{0\}) \cup_{\Lambda \times \{0\}} (\Lambda \times L),$$

a union of the **projection component** *and the* **cone components**.

If X is homogeneous, we get a coefficient-wise lower bound on the Hilbert series $h_X(s)$:

$$h_X(s) \geq h_\Pi(s) + \frac{s}{1-s} h_\Lambda(s)$$

with equality if and only if the set-theoretic equality for X' is also a scheme-theoretic equality.

Reducedness

When equality holds, we say that X' (or X) has a **geometric vertex decomposition**.

In practice, to check this equality, it is enough to show that X' is reduced.

Algebraically, we are studying the initial "y-forms" of the ideal $I \subseteq \mathbb{C}[x_1, \ldots, y = x_l, \ldots, x_n]$.

Matrix Schubert varieties

These may be presented as $B_- \times B_+$ orbit closures in $M_{n\times n}(\mathbb{C})$, or as cut out by determinants.

Given a permutation matrix $\pi \in S_n$ in $M_{n\times n}(\mathbb{C})$
$B_- = n \times n$ invertible lower triangular matrices
$B_+ = n \times n$ invertible upper triangular matrices

The **matrix Schubert variety** is

$\overline{X}_\pi = \overline{B_-\pi B_+}$, closure taken inside $M_{n\times n}(\mathbb{C})$.
[Fulton '96, Knutson-Miller '00]

It contains all matrices satisfying prescribed rank conditions, so determinantally, e.g.:

$$\overline{X}_{2143} := \mathrm{Spec}(\mathbb{C}[Z]/I_{2143})$$

where

$$I_{2143} := \langle z_{11}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} = z_{11}z_{22}z_{33} \pm \ldots \rangle$$

Equivariant cohomology classes

The torus $T \subset B_-$ acts on $M_{n\times n}(\mathbb{C})$ leaving $\overline{X}_\pi$ invariant.

Hence there is a T-equivariant cohomology class

$[\overline{X}_\pi]_T \in H_T^*(M_{n\times n}(\mathbb{C})) \cong H_T^*(\mathrm{pt}) \cong \mathbb{Z}[x_1,\ldots,x_n]$,

a *single polynomial* in $x_1,\ldots,x_n$ (more soon).

Equivalent geometry/topology/algebra

(1) (Classical algebraic geometry) Let (a_{ij}) for $1 \leq i \leq m$, $1 \leq j \leq n$ be a matrix of homogeneous forms in $x_0, x_1, \ldots x_N$. Fix a nonnegative integer r. Let

$$V_r = \{(x_0 : x_1 : \ldots : x_N) \in \mathbb{P}^N : \text{rank}(A) \leq r\}$$

What is the expected degree of V_r? (The case $r = 0$ is Bézout's theorem.)

(2) (Topology) Let X be a smooth algebraic variety (or manifold). Let $\Psi : E \to F$ be a morphism of vector bundles of rank m and n over X. Let

$$D_r(\Psi) = \{x \in X : \text{rank}(\Psi(x)) \leq r\}.$$

What is cohomology class $[D_r(\Psi)] \in H^*(X)$ in the Chern classes of E and F?

(3) (Commutative algebra) $\overline{X}_\pi = Spec(\mathbb{C}[Z]/I_\pi)$. What is the Hilbert series of $R = \mathbb{C}[Z]/I_\pi$?

In each case, connections to **Schur polynomials** were discovered.

Transition formulae

The **Schubert polynomial** is the equivariant Schubert class of $\overline{X}_\pi$:

$$[\overline{X}_\pi]_T = \mathfrak{S}_\pi(x_1, \ldots, x_n) \in H_T^*(M_{n\times n}(\mathbb{C})),$$

[Knutson-Miller '00, Fulton '96].

Let ℓ be the position of the last **descent** of π, i.e., where $\pi(\ell) > \pi(\ell+1)$, then the **Schubert transition formula** has the form:

$$\mathfrak{S}_\pi(x_1, \ldots, x_n) = \\ x_\ell S_{\pi'}(x_1, \ldots, x_n) + \sum_{\sigma \in \mathcal{T}(\pi)} \mathfrak{S}_\sigma(x_1, \ldots, x_n),$$

for some uniquely defined permutations $\pi', \mathcal{T}(\pi)$ [Lascoux and Schützenberger '82]

This formula uniquely *algebraically* determines the Schubert polynomials (differently than the *geometric* divided difference operators)

Goal: To give a *geometric* (degeneration) interpretation of the transition formula.

A geometric vertex decomposition for matrix Schubert varieties

Theorem 2 (Knutson-Y.) *Let* $\overline{X}_\pi$ *be any matrix Schubert variety. Let* ℓ *be the last descent of* π *and* $e = (\ell, \pi(\ell) - 1)$. *Let* L *be the matrices supported at* e, *and* H *those supported on the complement.*

Then X' *has a geometric vertex decomposition; it is reduced and Cohen-Macaulay. The projection component* $\Pi \times L$ *is a matrix Schubert variety and the divisor* $\Lambda \times L$ *is a union of several other matrix Schubert varieties. The resulting recursion for the Hilbert series of* $\overline{X}_\pi$ *is Lascoux's* K*-theory transition formula.*

Applications to geometry of combinatorial positivity

Theorem 2 gives a geometric framework that interprets many combinatorial numbers and constructions from the study of degeneracy loci and (K-theory, quantum) Schubert calculus, by counting components of a sequentially degenerated matrix Schubert variety.

<u>Some examples:</u>

(I) Monomial positivity of $\mathfrak{S}_\pi(x_1, \ldots, x_n)$, differently than in [Knutson-Miller '00]

(II) The classical Littlewood-Richardson rule.

(III) "Schur positivity" of Schubert polynomials [Buch-Kresch-Tamvakis-Y. '02]

(IV) Schur positivity of stable Schubert polynomials.

(V) The quiver coefficients [Buch-Fulton '99].

Another interpretation, K-theory, and new formulae

(VI) The Lascoux-Schützenberger transition tree is a tree of degenerations.

Our work extends to K-theory, explaining formulae for Grothendieck polynomials [Buch ‘02, Buch-Kresch-Tamvakis-Y. ‘03, Lascoux ‘01].

Moreover, the geometry naturally leads one to **new** formulae, see Allen Knutson’s talk, coming up next.

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