Math 5654 Midterm 1 Solutions, Spring 2009

Problem 1

By definition,

$$Q(C) = P(C|B) = \frac{P(BC)}{P(B)}.$$

Thus $P(BC) = 0$ if and only if $Q(C) = 0$. Hence in this case we have by the text's convention that $P(A|B\,C) = 0$ and that $Q(A|C) = 0$. Therefore the stated equality holds in this case.

It remains to consider the case that $P(BC) > 0$. In this case by definition we have

$$P(A|BC) = \frac{P(ABC)}{P(BC)}$$

and

$$Q(A|C) = \frac{Q(AC)}{Q(C)} = \frac{\frac{P(ACB)}{P(B)}}{\frac{P(CB)}{P(B)}}.$$

Thus again the stated equality holds, so the problem is finished.

Problem 2

(a)

$$P(X_0 = 1, X_1 = 2, X_2 = 3) = P(X_0 = 1)\, p(1,2)\, p(2,3)$$
$$= \tfrac{1}{3}\tfrac{1}{3}\tfrac{1}{2} = \frac{1}{18}.$$

Let π^n denote the distribution of X_n. Writing π^0 as a row vector, we are given that

$$\pi^0 = \begin{bmatrix} \frac{1}{3} & 0 & \frac{2}{3} \end{bmatrix}.$$

We have $\pi^1 = \pi^0 * p$. Hence, writing π^1 as a row vector and p as a matrix,

$$\pi^1 = \pi^0 p = \begin{bmatrix} \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}.$$

(b) From π^1, we have $P(X_1 = 3) = \frac{4}{9}$.

(c)

$$EX_1 = 1\pi^1(1) + 2\pi^1(2) + 3\pi^1(3) = \frac{21}{9} = \frac{7}{3}.$$

Problem 3

By Theorem 2.2.33, π^n converges to a limit distribution π.

Since

$$EX_n = \sum_{x\in S} x\pi^n(x),$$

$$EX_n \to \sum_{x\in S} x\pi(x) \text{ as } n\to\infty.$$

Problem 4

By definition, if (X_n) is a random walk then the sequence $X_0, X_1 - X_0, X_2 - X_1, \ldots$ must be independent. In particular X_0 and $X_1 - X_0$ must be independent.

$$P(X_0 = 1, X_1 - X_0 = 1) = P(X_0 = 1, X_1 = 2) = P(X_0 = 1)\, p(1,2) = \tfrac{1}{2}\tfrac{1}{3} = \tfrac{1}{6}.$$

Also

$$P(X_0 = -1, X_1 - X_0 = 1) = P(X_0 = -1, X_1 = 0) = P(X_0 = -1)\, p(-1,0) = \tfrac{1}{2}\tfrac{1}{2} = \tfrac{1}{4}.$$

Then

$$P(X_1 - X_0 = 1) = \tfrac{1}{6} + \tfrac{1}{4} = \frac{5}{12}.$$

Since

$$P(X_0 = 1, X_1 - X_0 = 1) \neq P(X_0 = 1)\, P(X_1 - X_0 = 1),$$

the events $\{X_1 - X_0 = 1\}$ and $\{X_0 = 1\}$ are not independent, and so the random variables $X_1 - X_0$ and X_0 are not independent.

Problem 5

(i) Let $x_0, \ldots, x_{n+1}, y_0, \ldots, y_{n+1} \in S$, such that

$$P(Z_0 = (x_0, y_0), \ldots, Z_n = (x_n, y_n)) > 0.$$

To check the Markov property we must show

$$P(Z_{n+1} = (x_{n+1}, y_{n+1}) \,|\, Z_0 = (x_0, y_0), \ldots, Z_n = (x_n, y_n)) = P(Z_{n+1} = (x_{n+1}, y_{n+1}) \,|\, Z_n = (x_n, y_n)).$$

We know that $Y_k = f(X_k)$ for all k. Since we assume that

$$P(Z_0 = (x_0, y_0), \ldots, Z_n = (x_n, y_n)) > 0,$$

it must be true that $y_k = f(x_k)$ for all $k = 0, \ldots, n$. We will use this fact in what follows. We have

$$\begin{aligned}
&P(Z_{n+1} = (x_{n+1}, y_{n+1}) | Z_0 = (x_0, y_0), \ldots, Z_n = (x_n, y_n)) = \\
&P(X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1} | X_0 = x_0, Y_0 = y_0, \ldots, X_n = x_n, Y_n = y_n) \\
&= P(X_{n+1} = x_{n+1}, f(x_{n+1}) = y_{n+1} | X_0 = x_0, f(x_0) = y_0, \ldots, X_n = x_n, f(x_n) = y_n) \\
&= P(X_{n+1} = x_{n+1}, f(x_{n+1}) = y_{n+1} | X_0 = x_0, \ldots, X_n = x_n) \\
&= \begin{cases} P(X_{n+1} = x_{n+1} | X_0 = x_0, \ldots, X_n = x_n) & \text{if } f(x_{n+1}) = y_{n+1}, \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} P(X_{n+1} = x_{n+1} | X_n = x_n) & \text{if } f(x_{n+1}) = y_{n+1}, \\ 0 & \text{otherwise} \end{cases} \\
&= P(X_{n+1} = x_{n+1}, f(x_{n+1}) = y_{n+1} | X_n = x_n) \\
&= P(X_{n+1} = x_{n+1}, f(x_{n+1}) = y_{n+1} | X_n = x_n, f(x_n) = y_n) \\
&= P(X_{n+1} = x_{n+1}, Y_{n+1} = y_{n+1} | X_n = x_n, Y_n = y_n).
\end{aligned}$$

This proves the Markov property.

(ii) The chain need not be time-homogeneous because (X_n) need not be time-homogeneous.

(iii) Consider the sets $A_{0,1,2}, A_{0,1,0}, A_{0,-1,0}, A_{0,-1,-2}$, where $A_{0,1,2}$ is the event $\{X_0 = 0, X_1 = 1, X_2 = 2\}$ and so on. Each of these events has probability $1/4$.

We notice that the values of Y_0, Y_1, Y_2 are $1, 1, 1$ on $A_{0,1,2}$, $1, 1, 1$ on $A_{0,1,0}$, $1, -1, 1$ on $A_{0,-1,0}$ and $1, -1, -1$ on $A_{0,-1,-2}$.

After studying these, we consider

$$P(Y_3 = -1 | Y_0 = 1, Y_1 = -1, Y_2 = 1).$$

We have shown that $P(Y_0 = 1, Y_1 = -1, Y_2 = 1) = P(A_{0,-1,0}) = 1/4$. Similarly we see that $\{Y_0 = 1, Y_1 = -1, Y_2 = 1, Y_3 = -1\}$ is the event $A_{0,-1,0,-1}$, and has probability $1/8$. Hence

$$P(Y_3 = -1 | Y_0 = 1, Y_1 = -1, Y_2 = 1) = \frac{\frac{1}{8}}{\frac{1}{4}} = \frac{1}{2}.$$

On the other hand,

$$P\left(Y_2 = 1\right) = P\left(A_{0,1,2}\right) + P\left(A_{0,1,0}\right) + P\left(A_{0,-1,0}\right) = \frac{3}{4}.$$

Also

$$P\left(Y_2 = 1, Y_3 = -1\right) = P\left(A_{0,1,0,-1}\right) + P\left(A_{0,-1,0,-1}\right) = \frac{1}{4}.$$

Hence

$$P\left(Y_3 = -1 | Y_2 = 1\right) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Thus (Y_n) does not satisfy the Markov property.

Problem 6

(i) The random walk must exit from the interval (a, b) either at a or at b. Hence $1 - u(x)$ is the probability to reach b before reaching a, starting from x.

(ii) For each integer x with $a \leq x \leq b$, let $w(x) = u(x) + (1 - u(x))v(b)$. We can check easily that w and v satisfy the same equation on (a, b) and the same boundary conditions at a and b. By Lemma 2.1.15, they must be equal on $[a, b]$, so the problem is finished.

We can also think about calculating $v(x)$ by following the random walk until it exits from (a, b). It exits at a with probability $u(x)$, by the definition of u. If it exits at b, we don't know whether it will reach a or c first. But since the random walk starts over after it reaches a point, the chance it will reach a before c, given that it reached b before a, is exactly $v(b)$. Hence

$$v(x) = u(x) + P\left(\text{reach } a \text{ before } c \mid \text{reach } b \text{ before } a\right) P\left(\text{ reach } b \text{ before } a\right),$$

that is, $v(x) = u(x) + v(b)\left(1 - u(x)\right)$.