Math 5654 Midterm 2 Solutions

Problem 1

Let $Z = \Pi_{\mathcal{L}} X$.

We have $X = cY$, where $X - cY \perp Y$. Then $\langle X, Y\rangle - c\langle Y, Y\rangle = 0$, so $3 - c4 = 0$, or $c = 3/4$.

Problem 2

Let $Y = (Y_1, Y_2)$.

(i)

$$\Sigma_{Y,Y} = \begin{bmatrix} \mathrm{Cov}\,(Y_1, Y_1) & \mathrm{Cov}\,(Y_1, Y_2) \\ \mathrm{Cov}\,(Y_2, Y_1) & \mathrm{Cov}\,(Y_2, Y_2) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

(ii) By Problem 4.3.8 in the text,

$$a_1 Y_1 + a_2 Y_2 = AY,$$

where

$$A = Q_{XY} Q_{YY}^{-1} = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}.$$

Thus $a_1 = 2$ and $a_2 = -1$.

More direct approach We can solve two equations: $X - a_1 Y_1 - a_2 Y_2 \perp Y_1$, $X - aY_1 - a_2 Y_2 \perp Y_2$, which gives us

$$\begin{cases} 3 - 2a_1 - a_2 & = & 0, \\ 1 - a_1 - a_2 & = & 0. \end{cases}$$

Problem 3

By Definition 4.2.18, every linear combination of $X_1, \ldots, X_m$ is a one-dimensional Gaussian random variable.

Since each element Y_j is a linear combination of $X_1, \ldots, X_m$, every linear combination of $Y_1, \ldots, Y_n$ is a linear combination of $X_1, \ldots, X_m$. Thus every linear combination of $Y_1, \ldots, Y_n$ is a one-dimensional Gaussian random variable.

By Definition 4.2.18, $(Y_1, \ldots, Y_n)$ is Gaussian.

Problem 4

(i) Since $\mathcal{L}_1 \subset \mathcal{L}_2$, $Z \in \mathcal{L}_2$. So Z cannot be a better approximation than W, the best approximation in $\mathcal{L}_2$. That is, $\|X - W\| \leq \|X - Z\|$.

(ii) When (X, Y_1, Y_2) is Gaussian, we know by Theorem 4.2.25 that

$$E\left(X|Y_1, Y_2\right) = Z.$$

Thus Z is the best approximation to X in all of $\sigma\left(Y_1, Y_2\right)$. Since $\mathcal{L}_2 \subset \sigma\left(Y_1, Y_2\right)$, Z is the best approximation to X in $\mathcal{L}_2$. That is, $Z = W$.

Problem 5

Consider a nonnegative function f on $\mathbb{R}^{n+2}$.

Then

$$\begin{aligned}
&Ef\left(X_0, \ldots, X_{n+1}\right) \\
&= \int f\left(x_0, \ldots, x_{n+1}\right) \pi_0\left(x_0\right) p\left(x_0, x_1\right) \ldots p\left(x_{n-1}, x_n\right) p\left(x_n, x_{n+1}\right)\, dx_0 \,\ldots\, dx_{n+1} \\
&\qquad = \int_{\mathbb{R}^{n+1}} \left(\int_{\mathbb{R}} f\left(x_0, \ldots, x_n, x_{n+1}\right) p\left(x_n, x_{n+1}\right)\, dx_{n+1}\right) \\
&\qquad\qquad \pi_0\left(x_0\right) p\left(x_0, x_1\right) \ldots p\left(x_{n-1}, x_n\right)\, dx_0 \,\ldots\, dx_n \\
&\qquad\qquad = E\left(\int_{\mathbb{R}} f\left(X_0, \ldots, X_n, x_{n+1}\right)) p\left(X_n, x_{n+1}\right)\, dx_{n+1}\right).
\end{aligned}$$

By Definition 5.1.2, (X_n) is a time-homogeneous Markov chain with transition kernel p.

Problem 6

(i) When h is larger, the *relative* size of the noise is smaller.

(ii) In the notation of Problem 5.2.3, $v^2 = h^2 + 1$.

By Problem 5.2.3, the limit of $\bar{\sigma}_n^2$ is given by x, where x is the unique *positive* solution of

$$x = \frac{1}{h^2+1}\frac{x}{h^2 x + h^2 + 1} + \frac{1}{h^2+1}.$$

Simplifying, we have

$$\left(h^2+1\right) x = \frac{x}{h^2 x + h^2 + 1} + 1.$$

This gives

$$h^2\left(h^2+1\right)x^2 + \left(h^2+1\right)^2 x = x + h^2 x + h^2 + 1.$$

Dividing both sides by h^2+1 we have

$$h^2 x^2 + \left(h^2+1\right) x = x + 1,$$

or

$$h^2 x^2 + h^2 x - 1 = 0,$$

i.e.

$$x^2 + x - \frac{1}{h^2} = 0.$$

Hence

$$x = \frac{-1 + \sqrt{1 + \frac{4}{h^4}}}{2}.$$

This quantity decreases as h increases, which is consistent with (i).

Problem 7

First, a remark. For any nonnegative function h on $\mathbb{R}^{n+1}$,

$$Eh(X_0,\ldots,X_n) = Eh(X_0, X_0U_1,\ldots,X_0U_1\ldots U_n)$$
$$= \int_{\mathbb{R}\times[0,1]^n} f(x_0, x_0u_1,\ldots,x_0u_1\ldots u_n)\,\pi_0(x_0)\;dx_0\,du_1\ldots\,du_n.$$

Now consider a nonnegative function f on $\mathbb{R}^{n+2}$.
Then

$$Ef(X_0,\ldots,X_{n+1}) = Ef(X_0, X_0U_1,\ldots,X_0U_1\ldots U_{n+1})$$
$$= \int_{\mathbb{R}\times[0,1]^{n+1}} f(x_0, x_0u_1,\ldots,x_0u_1\ldots u_{n+1})\,\pi_0(x_0)\;dx_0\,du_1\ldots\,du_{n+1}$$
$$= \int_{\mathbb{R}\times[0,1]^n}\left(\int_{[0,1]} f(x_0, x_0u_1,\ldots,x_0u_1\ldots u_{n+1})\;du_{n+1}\right)\pi_0(x_0)\;dx_0\,du_1\ldots\,du_n.$$

Now we apply the remark, with

$$h(x_0,\ldots,x_n) = \int_{[0,1]} f(x_0, x_0u_1,\ldots,x_0u_1\ldots u_{n+1})\;du_{n+1},$$

and we see that

$$Ef(X_0,\ldots,X_{n+1}) = E\left(\int_{[0,1]} f(X_0, X_1,\ldots,X_n, X_nu_{n+1})\;du_{n+1}\right).$$

Making the substitution $x_nu_{n+1} = x_{n+1}$, $X_ndu_{n+1} = dx_{n+1}$, this becomes

$$Ef(X_0,\ldots,X_{n+1}) = E\left(\int_{[0,X_n]} f(X_0, X_1,\ldots,X_n, x_{n+1})\;\frac{1}{X_n}dx_{n+1}\right)$$
$$= E\left(\int_{\mathbb{R}} f(X_0, X_1,\ldots,X_n, x_{n+1})\;\frac{1}{X_n}I_{[0,X_n]}(x_{n+1})\,dx_{n+1}\right).$$

By Definition 5.1.2, (X_n) is a time-homogeneous Markov chain with transition kernel p, where

$$p(x,y) = \frac{1}{x}I_{[0,x]}(y).$$

Alternate approach used by one student Let $r(y) = I_{[0,1]}(y)$, so that r is the density of U_n for each n.

Let $\rho_n(x_0, \ldots, x_n)$ be the density of $(X_0, \ldots, X_n)$, assuming that it exists. Then the joint density of $(X_0, \ldots, X_n, U_{n+1})$ is $\rho_n(x_0, \ldots, x_n)\, r(u_{n+1})$.

For any nonnegative function f on $\mathbb{R}^{n+2}$,

$$
\begin{aligned}
&Ef(X_0, X_1, \ldots, X_{n+1}) = Ef(X_0, X_1, \ldots, X_n, X_n U_{n+1}) \\
&\quad = \int f(x_0, \ldots, x_n, x_n u_{n+1})\, \rho(x_0, \ldots, x_n)\, r(u_{n+1})\; dx_0\; \ldots,\; dx_n\, du_{n+1} \\
&= \int f(x_0, \ldots, x_n, x_n u_{n+1})\, \rho(x_0, \ldots, x_n)\, r\left(\tfrac{x_{n+1}}{x_n}\right)\; dx_0\; \ldots,\; dx_n\; \left(\tfrac{1}{x_n}\right) dx_{n+1} \\
&= \int f(x_0, \ldots, x_n, x_n u_{n+1})\, \rho(x_0, \ldots, x_n)\, p(x_n, x_{n+1})\; dx_0\; \ldots,\; dx_n\, dx_{n+1},
\end{aligned}
$$

where $p(x_n, x_{n+1}) = \left(\frac{1}{x_n}\right) r\left(\frac{x_{n+1}}{x_n}\right)$. Hence

$$
\rho_{n+1}(x_0, \ldots, x_{n+1}) = \rho_n(x_0, \ldots, x_n)\, p(x_n, x_{n+1}).
$$

Considering $n = 0, 1, \ldots$, it follows that $\rho_n(x_0, \ldots, x_n)$ exists for all n and is given by

$$
\rho_n(x_0, \ldots, x_n) = \pi_0(x_0)\, p(x_0, x_1) \ldots p(x_{n-1}, x_n).
$$

Thus Problem 5 of the exam is applicable.