Discrete-time Filtering

1 The one-dimensional case

We are working on Section 5.2:1 here. By Problem 5.1.4, all the random vectors we encounter will have Gaussian densities.

We want to study $\pi_n\left(x|y_{[0,n]}\right)$. By definition, this is the conditional density of X_n given $Y_{[0,n]} = y_{[0,n]}$.

As in the text let $\bar{m}_n = E\left[X_n|\,Y_0,\ldots,Y_n\right]$. There is some function $\varphi_n(y_0,\ldots,y_n)$ such that $\bar{m}_n = \varphi_n\left(Y_0,\ldots,Y_n\right)$.

By Problem 4.3.26, one possible choice for $\pi_n\left(x|y_{[0,n]}\right)$ is the density with mean $\varphi_n\left(y_0,\ldots,y_n\right)$ and some variance $\bar{\sigma}_n^2$ which does not depend on $y_0,\ldots,y_n$. We will often just write φ_n instead of $\varphi_n\left(y_0,\ldots,y_n\right)$.

Equation (4) of Theorem 5.1.8 (the filtering equation) says that

$$\pi_{n+1}\left(x|y_{[0,n+1]}\right) = \frac{1}{\xi_{n+1}\left(y_{[0,n+1]}\right)}\int_{\mathbb{R}} \pi_n\left(r|y_{[0,n]}\right) q^{rx}\left(y_n,y_{n+1}\right)\,dr. \tag{1}$$

Substituting for $\pi_n\left(x|y_{[0,n]}\right)$ and using the formula for the transition kernel, we have

$$\pi_{n+1}\left(x|y_{[0,n+1]}\right) = \frac{1}{\xi_{n+1}\left(y_{[0,n+1]}\right)}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\bar{\sigma}_n} e^{-\frac{1}{2\bar{\sigma}_n^2}(r-\varphi_n)^2}\frac{1}{2\pi cg} e^{-\frac{1}{2c^2}(x-ar-b)^2} e^{-\frac{1}{2g^2}(y_{n+1}-hx)^2}\,dr.$$

That is,

$$\pi_{n+1}\left(x|y_{[0,n+1]}\right) = \frac{e^{-\frac{1}{2g^2}(y_{n+1}-hx)^2}}{\sqrt{2\pi}ga\xi_{n+1}\left(y_{[0,n+1]}\right)}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\bar{\sigma}_n} e^{-\frac{1}{2\bar{\sigma}_n^2}(r-\varphi_n)^2}\frac{1}{\sqrt{2\pi}\frac{c}{a}} e^{-\frac{1}{2\left(\frac{c}{a}\right)^2}\left(\frac{x-b}{a}-r\right)^2}\,dr.$$

Now we can recognise the integral as the convolution of two normal densities. To make it clearer, let $u = r-\varphi_n$. Then our equation becomes

$$\pi_{n+1}\left(x|y_{[0,n+1]}\right) = \frac{e^{-\frac{1}{2g^2}(y_{n+1}-hx)^2}}{\sqrt{2\pi}ga\xi_{n+1}\left(y_{[0,n+1]}\right)}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}\bar{\sigma}_n} e^{-\frac{1}{2\bar{\sigma}_n^2}u^2}\frac{1}{\sqrt{2\pi}\frac{c}{a}} e^{-\frac{1}{2\left(\frac{c}{a}\right)^2}\left(\frac{x-b}{a}-\varphi_n-u\right)^2}\,du.$$

In class we proved that the convolution of two normal densities is again a normal density. (And of course the mean and variance of the resulting density is determined in the usual way from the means and variances of the factors in the convolution.) Hence we obtain

$$\pi_{n+1}\left(x|y_{[0,n+1]}\right) = \frac{e^{-\frac{1}{2g^2}(y_{n+1}-hx)^2}}{\sqrt{2\pi}ga\xi_{n+1}\left(y_{[0,n+1]}\right)} \frac{1}{\sqrt{2\pi\left(\bar{\sigma}_n^2 + \left(\frac{c}{a}\right)^2\right)}} e^{-\frac{1}{2\left(\bar{\sigma}_n^2+\left(\frac{c}{a}\right)^2\right)}\left(\frac{x-b}{a}-\varphi_n\right)^2}. \tag{2}$$

By definition we also have

$$\pi_{n+1}\left(x|y_{[0,n+1]}\right) = \frac{1}{\sqrt{2\pi}\bar{\sigma}_{n+1}} e^{-\frac{1}{2\bar{\sigma}_{n+1}^2}(x-\varphi_{n+1})^2}. \tag{3}$$

Hence we can obtain expressions for $\bar{\sigma}_{n+1}$ and φ_{n+1} by simply comparing (3) and (2).

Comparing the coefficients of x^2 in the exponents shows us that

$$\frac{1}{\bar{\sigma}_{n+1}^2} = \frac{h^2}{g^2} + \frac{1}{a^2\bar{\sigma}_n^2 + c^2}. \tag{4}$$

Comparing the coefficients of x in the exponents shows us that

$$\frac{\varphi_{n+1}}{\bar{\sigma}_{n+1}^2} = \frac{hy_{n+1}}{g^2} + \frac{a\varphi_n + b}{a^2\bar{\sigma}_n^2 + c^2}. \tag{5}$$

Once we have φ_{n+1} and $\bar{\sigma}_{n+1}$ we can find ξ_{n+1} if it is ever needed.

It is routine to check that (4) and (5) are equivalent to equations (4) and (5) in Section 5.2:1 on page 135 of the text.

2 Introduction to the multidimensional case

The algebra in this note applies to the Gaussian filtering situation in Section 2 of Chapter 5 and also applies to the linear filtering situation in Section 3 of Chapter 5.

To avoid ambiguity, for any $\mathbb{R}^m$-valued random variable U, in what follows we will write the coordinates of U using *superscripts*, so that $U = \left(U^1, \ldots, U^m\right)$.

We will assume that we are given random vectors $X_n, Y_n, M_n, R_n,\ n = 0, 1, \ldots$, as follows. X_n takes values in $\mathbb{R}^d$, Y_n takes values in $\mathbb{R}^{d_1}$, M_n takes values in $\mathbb{R}^d$, R_n takes values in $\mathbb{R}^{d_1}$. All coordinates of all these random vectors are random variables in L_2.

In the Gaussian case (Section 5.2:2), we have $M_n = PW_n$ and $R_n = QW_n$.

We assume that for each $n = 0, 1, \ldots$ there exist constant matrices A_n, H_n, such that

$$X_{n+1} = A_n X_n + M_n,\ Y_n = H_n X_n + R_n, \tag{6}$$

for all $n = 0, 1, \ldots$.

We assume for each $n = 0, 1, \ldots$ that all coordinate random variables of M_n and R_n have mean zero. That is, we assume that M_n and R_n are mean-zero random vectors.

We also assume that the sequence

$$X_0, (M_0, R_0), (M_1, R_1), (M_2, R_2), \ldots$$

is uncorrelated. (We do not assume that M_n, R_n are uncorrelated.)

These assumptions are certainly true in the situation of Section 4.2:2 because we assume there that the random vectors $X_0, W_0, W_1, \ldots$ are independent.

Let

$$\mathcal{L}_n = \mathrm{Span}\left(1, Y_{[0,n]}\right), \tag{7}$$

for each $n = 0, 1, 2, \ldots$. Let $\mathcal{L}_{-1} = \mathrm{Span}\,(1)$.

Notation In this note for brevity we use Span in a *special sense*, so that $\mathrm{Span}\left(1, Y_{[0,n]}\right)$ denotes the span of the set of *all* random variables Y_j^i, $i = 1, \ldots, d_1$, $j = 0, 1, \ldots, n$, together with the constant random variable equal to 1. (Recall that we use superscripts here to indicate the coordinates of a random vector.) $\mathcal{L}_n$ is a finite-dimensional subspace of L_2. The members of $\mathcal{L}_n$ are scalar-valued random variables.

Notation In the same spirit, in this note for brevity we write $U \perp \mathcal{L}_n$ when U is an $\mathbb{R}^m$-valued random variable to mean that $U^i \perp \mathcal{L}_n$ for each $i = 1, \ldots, m$.

Let the projection Π_n be defined by

$$\Pi_n = \Pi_{\mathcal{L}_n}, \tag{8}$$

for $n = -1, 0, 1, 2, \ldots$. Let

$$\bar{X}_n = \Pi_{n-1} X_n, \; \bar{Y}_n = \Pi_{n-1} Y_n, \tag{9}$$

for $n = 0, 1, \ldots$. We note that $\bar{X}_n$ is denoted by $\bar{X}_{n|n-1}$ in the Gaussian case treated in Section 5.2:2, and $\bar{X}_n$ is denoted by $\hat{X}_{n|n-1}$ in the linear filtering case treated in Section 5.3.

Notation It is important that in (9) we are applying Π_n coordinatewise, i.e. in the simplest possible way.

For any real-valued random variables $U, V \in L_2$, $\mathrm{Cov}\,(U, V) = EUV - EUEV$. Hence if U, V are uncorrelated and *at least one* of U, V has mean zero, then $U \perp V$.

It follows from (6) by an easy induction (using the mean-zero assumption for M_{n+1} and R_{n+1} and the uncorrelated property) that:

(i) every coordinate function of M_{n+1} and R_{n+1} is orthogonal to $\mathcal{L}_n$, and

(ii) every coordinate function of M_n and R_n is also orthogonal to every coordinate function of X_n.

In particular we have

$$\Pi_n M_{n+1} = 0, \ \Pi_n R_{n+1} = 0. \tag{10}$$

Note that the value of $\bar{X}_{n+1}$ is known once the values of $Y_0, \ldots, Y_n$ are known.

More specifically, since $\bar{X}_{n+1} \in \mathcal{L}_n$ we know that by definition the following holds. There are constants t_i, for each $i = 1, \ldots, d$, and $c_0^{ij}, \ldots, c_n^{ij}$, for each $i = 1, \ldots, d$, and each $j = 1, \ldots, d_1$, such that

$$\bar{X}_{n+1}^i = t_i + \sum_{j=1}^{d_1} c_0^{ij} Y_0^j + \ldots + \sum_{j=1}^{d_1} c_n^{ij} Y_n^j.$$

Note that all these constants depend on n, but we suppress that to avoid clutter. So we should think of n as having some fixed but arbitrary value in what follows.

Let $t = (t_1, \ldots, t_d)$, and let $C_\ell = \left(c_\ell^{ij} \right)$. Then

$$\bar{X}_{n+1} = t + C_0 Y_0 + \ldots + C_n Y_n. \tag{11}$$

Since $X_{n+1}^i - \bar{X}_{n+1}^i \perp 1$ for each i, we know that $X_{n+1}^i - \bar{X}_{n+1}^i$ has mean zero for each i, that is, $X_{n+1} - \bar{X}_{n+1}$ is a mean-zero vector. Similarly $Y_n - \bar{Y}_n$ is a mean-zero vector.

For convenience, let

$$\overline{\Sigma}_{n+1} = \Sigma_{X_{n+1} - \bar{X}_{n+1}, X_{n+1} - \bar{X}_{n+1}}. \tag{12}$$

As the text notes, $\overline{\Sigma}_{n+1}$ is nonrandom, so it can be computed offline.

Our main task will be to derive a useful recursive formula for $\bar{X}_{n+1}$ in terms of $\bar{X}_n$.

3 Basic relations

Since $\mathcal{L}_{n-1} \subset \mathcal{L}_n$, we see from the definition of projection that

$$\Pi_{n-1} \Pi_n = \Pi_{n-1}. \tag{13}$$

This is the *telescoping* property for orthgonal projections, and we used this to establish the similar telescoping property for conditional expectation.

From linearity and orthogonality we also have

$$\Pi_{n-1} X_{n+1} = A_n \bar{X}_n, \ \bar{Y}_n = H_n \bar{X}_n. \tag{14}$$

It follows from (13) and (14) that

$$\Pi_{n-1} \bar{X}_{n+1} = A_n \bar{X}_n. \tag{15}$$

Substituting into (15) from (11),

$$t + C_0 Y_0 + \ldots + C_{n-1} Y_{n-1} + C_n \bar{Y}_n = A_n \bar{X}_n. \tag{16}$$

Hence

$$\bar{X}_{n+1} = A_n \bar{X}_n + C_n \left(Y_n - \bar{Y}_n \right). \tag{17}$$

Equation (17) is not quite the recursive equation we want, since we do not know C_n yet.

From (17) and (6) we have

$$X_{n+1} - \bar{X}_{n+1} = A_n \left(X_n - \bar{X}_n \right) + M_n - C_n \left(Y_n - \bar{Y}_n \right). \tag{18}$$

From (14) and (6),

$$Y_n - \bar{Y}_n = H_n \left(X_n - \bar{X}_n \right) + R_n. \tag{19}$$

Since $X_n - \bar{X}_n$ and R_n are uncorrelated, we have

$$\Sigma_{Y_n - \bar{Y}_n, Y_n - \bar{Y}_n} = H_n \overline{\Sigma}_n H_n^* + \Sigma_{R_n, R_n}. \tag{20}$$

Since every coordinate of $X_{n+1} - \bar{X}_{n+1}$ is orthogonal to $\mathcal{L}_n$, it follows that

$$\Sigma_{X_{n+1} - \bar{X}_{n+1}, Y_n - \bar{Y}_n} = 0. \tag{21}$$

Applying this fact to (18) then gives

$$0 = \Sigma_{A_n \left(X_n - \bar{X}_n \right) M_n - C_n \left(Y_n - \bar{Y}_n \right), Y_n - \bar{Y}_n}, \tag{22}$$

that is,

$$C_n \Sigma_{Y_n - \bar{Y}_n, Y_n - \bar{Y}_n} = \Sigma_{A_n \left(X_n - \bar{X}_n \right) + M_n, Y_n - \bar{Y}_n}. \tag{23}$$

Using (19),

$$C_n \Sigma_{Y_n - \bar{Y}_n, Y_n - \bar{Y}_n} = \Sigma_{A_n \left(X_n - \bar{X}_n \right) + M_n, H_n \left(X_n - \bar{X}_n \right) + R_n}. \tag{24}$$

Since $X_n - \bar{X}_n, M_n$ are uncorrelated and $X_n - \bar{X}_n, R_n$ are uncorrelated, we obtain

$$C_n \Sigma_{Y_n - \bar{Y}_n, Y_n - \bar{Y}_n} = A_n \overline{\Sigma}_n H_n^* + \Sigma_{M_n, R_n}. \tag{25}$$

4 The nonsingular special case

At this point, suppose that we add the assumption that

$$H_n \overline{\Sigma}_n H_n^* + \Sigma_{R_n, R_n} \text{ is nonsingular.}$$

Since $H_n\overline{\Sigma}_n H_n^*$ is nonnegative definite, this assumption will certainly hold if Σ_{R_n,R_n} is positive definite, since then we have that

$$H_n\overline{\Sigma}_n H_n^* + \Sigma_{R_n,R_n} \text{ is positive definite.}$$

Under this assumption we find using (20) that

$$C_n = \left(A_n\overline{\Sigma}_n H_n^* + \Sigma_{M_n,R_n}\right)\left(H_n\overline{\Sigma}_n H_n^* + \Sigma_{R_n,R_n}\right)^{-1}. \tag{26}$$

With this value for C_n, we can then obtain a satisfactory recursion formula from (17), substituting from (14) to eliminate $\bar{Y}_n$. We obtain

$$\bar{X}_{n+1} = A_n\bar{X}_n + C_n\left(Y_n - H_n\bar{X}_n\right) \tag{27}$$

with C_n given by (26). This gives equation (12) of Theorem 5.2.8.

5 Solving for C_n in the general case

Now we drop any assumption of nonsingularity, so we are in the general case.

We have shown in the general case that there exists a matrix C_n of constants such that (17) holds and hence that (27) holds. In general C_n is not uniquely determined by the condition (11), but at least one suitable value exists.

Our problem is to find a value for C_n, or at least to explain how to compute it. The *existence* of C_n is not in doubt.

As a consequence of (17), we know C_n satisfies (25). In general this equation will have many solutions. So we can say that the value of C_n is one of the possible solutions G of the equation

$$G\Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n} = A_n\overline{\Sigma}_n H_n^* + \Sigma_{M_n,R_n}. \tag{28}$$

Let G be any solution of (28), and let L also be any solution. Then

$$(G-L)\,\Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n} = 0. \tag{29}$$

Hence we have

$$(G-L)\,\Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n}\,(G-L)^* = 0, \tag{30}$$

or equivalently

$$\operatorname{Cov}\left((G-L)\left(Y_n-\bar{Y}_n\right),(G-L)\left(Y_n-\bar{Y}_n\right)\right) = 0. \tag{31}$$

Thus $(G-L)\left(Y_n-\bar{Y}_n\right)$ is constant almost surely.

Since $Y_n-\bar{Y}_n$ is a mean-zero vector, $(G-L)\left(Y_n-\bar{Y}_n\right)$ is a mean-zero vector. We conclude that (a.s.)

$$(G-L)\left(Y_n-\bar{Y}_n\right) = 0. \tag{32}$$

We know that there is one solution G of (28) such that

$$\bar{X}_{n+1} = A_n \bar{X}_n + G\left(Y_n - \bar{Y}_n\right). \tag{33}$$

holds, namely the solution $G = C_n$, using any correct value of C_n. But by (32) it then follows that (33) holds for *every* solution G of (28).

This gives us a convenient way to obtain our recursive equation in the general case. Take *any* solution G of (28) (which we know has a solution) using any method, say row-reduction, to find this solution. Then substitute this value for C_n in (27) to get an explicit recursive formula.

6 A recursive equation for $\overline{\Sigma}_n$

A recursive equation for $\overline{\Sigma}_n$ follows directly from (27). Indeed, using (6) and (17) we have

$$\begin{aligned} X_{n+1} - \bar{X}_{n+1} &= A_n\left(X_n - \bar{X}_n\right) + M_n - C_n\left(Y_n - H_n\bar{X}_n\right) \\ &= A_n\left(X_n - \bar{X}_n\right) + M_n - C_n\left(H_n\left(X_n - \bar{X}_n\right) + R_n\right) \\ &= \left(A_n - C_nH_n\right)\left(X_n - \bar{X}_n\right) + M_n - C_nR_n. \end{aligned}$$

As remarked earlier, every coordinate function of M_{n+1} and R_{n+1} is orthogonal to $\mathcal{L}_n$, and every coordinate function of M_n and R_n is also orthogonal to every coordinate function of X_n. Hence $(A_n - C_nH_n)\left(X_n - \bar{X}_n\right)$ and $M_n - C_nR_n$ are uncorrelated. Thus

$$\begin{aligned} \overline{\Sigma}_{n+1} &= \Sigma_{(A_n - C_nH_n)\left(X_n - \bar{X}_n\right),(A_n - C_nH_n)\left(X_n - \bar{X}_n\right)} \\ &+ \Sigma_{M_n - C_nR_n, M_n - C_nR_n} \\ &= (A_n - C_nH_n)\,\overline{\Sigma}_n\,(A_n - C_nH_n)^* \\ &+ \Sigma_{M_n - C_nR_n, M_n - C_nR_n}. \end{aligned} \tag{34}$$

This gives equation (13) of Theorem 5.2.8.

7 Projecting on parts of a space

It may be worthwhile to look a little more abstractly at the derivation of (17).

Let U be *any* random vector taking values in $\mathbb{R}^m$, such that all components of U are in L^2.

Fix n.

Using the definition just as in the derivation of (11), we can show that there exists a constant vector $\tau \in \mathbb{R}^m$ and constant matrices $\gamma_0, \ldots, \gamma_n$ of size $m \times d_1$ such that

$$\Pi_n U = \tau + \gamma_0 Y_0 + \ldots + \gamma_n Y_n. \tag{35}$$

(The constants depend on n but we are fixing n for the discussion.)

By telescoping we have

$$\Pi_{n-1}U = \Pi_{n-1}\Pi_n U. \tag{36}$$

Substituting for $\Pi_n U$ gives us

$$\Pi_{n-1}U = \tau + \gamma_0 Y_0 + \ldots + \gamma_{n-1}Y_{n-1} + \gamma_n \bar{Y}_n. \tag{37}$$

Hence

$$\Pi_n U = \Pi_{n-1}U + \gamma_n \left(Y_n - \bar{Y}_n\right). \tag{38}$$

Recall that by definition we have $Y_n - \bar{Y}_n \perp \mathcal{L}_{n-1}$. Let $\mathcal{W}_n = \mathrm{Span}\left(Y_n - \bar{Y}_n\right)$, so that $\mathcal{W}_n \perp \mathcal{L}_{n-1}$.

Equation (38) shows us that any vector in $\mathcal{L}_n$ can be written as a sum of a vector in $\mathcal{L}_{n-1}$ and a vector in $\mathcal{W}_n$, and it also shows us that

$$\Pi_n = \Pi_{n-1} + \Pi_{\mathcal{W}_n}. \tag{39}$$

The same result holds in general, whenever we can write a space as the sum of two orthogonal subspaces. Another expression of this idea is given in Problems 4.1.24.

8 Studying $\Pi_n X_n$

Let

$$\tilde{X}_n = \Pi_n X_n.$$

Similarly let

$$\tilde{\Sigma}_n = \Sigma_{X_n - \tilde{X}_n, X_n - \tilde{X}_n}.$$

For filtering purposes, it is more natural to consider $\tilde{X}_n$ than $\bar{X}_n$.

We would like to relate $\tilde{X}_n$ and $\tilde{\Sigma}_n$ to $\bar{X}_n$ and $\overline{\Sigma}_n$.

Fix n.

By (38) we have

$$\tilde{X}_n = \bar{X}_n + \gamma_n \left(Y_n - \bar{Y}_n\right), \tag{40}$$

for some constant matrix γ_n. As usual we would like to evaluate the constant, in this case γ_n. We have

$$X_n - \tilde{X}_n = X_n - \bar{X}_n - \gamma_n \left(Y_n - \bar{Y}_n\right). \tag{41}$$

Since $X_n - \tilde{X}_n \perp \mathcal{L}_n$ and consequently $X_n - \tilde{X}_n$ is mean-zero we also have

$$\Sigma_{X_n - \tilde{X}_n, Y_n - \bar{Y}_n} = 0. \tag{42}$$

Hence

$$0 = \Sigma_{X_n - \bar{X}_n, Y_n - \bar{Y}_n} - \Sigma_{\gamma_n\left(Y_n - \bar{Y}_n\right), \left(Y_n - \bar{Y}_n\right)}. \tag{43}$$

That is,

$$\Sigma_{\gamma_n(Y_n-\bar{Y}_n),Y_n-\bar{Y}_n} = \Sigma_{X_n-\bar{X}_n,Y_n-\bar{Y}_n}. \tag{44}$$

Hence

$$\gamma_n \Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n} = \Sigma_{X_n-\bar{X}_n,H_n(X_n-\bar{X}_n)+R_n} \tag{45}$$

and so

$$\gamma_n \Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n} = \overline{\Sigma}_n H_n^*. \tag{46}$$

We showed earlier that if G and L are matrices such that (29) holds then also (32) holds. This applies to the present situation, and shows that for *any* solutions γ_n of (46) we have the desired equation (40). Thus even when $\Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n}$ is singular, we can compute any solution for (46) we like, by any method we like, since it will then satisfy (40).

Assume that we are in the *nonsingular* case. Using (20) we have

$$\gamma_n = \overline{\Sigma}_n H_n^* \left(H_n \overline{\Sigma}_n H_n^* + \Sigma_{R_n,R_n}\right)^{-1}. \tag{47}$$

Once we have (47) we can substitute for γ_n in (40), use (14) to get rid of $\bar{Y}_n$, and obtain

$$\tilde{X}_n = \bar{X}_n + \overline{\Sigma}_n H_n^* \left(H_n \overline{\Sigma}_n H_n^* + \Sigma_{R_n,R_n}\right)^{-1} \left(Y_n - H_n \bar{X}_n.\right). \tag{48}$$

This gives the first equation of Problem 5.2.18.

We also need to consider $\tilde{\Sigma}_n$. By rearranging (41) we have

$$X_n - \tilde{X}_n + \gamma_n \left(Y_n - \bar{Y}_n\right) = X_n - \bar{X}_n. \tag{49}$$

Writing down the covariance matrix for each side using (42) we then obtain

$$\tilde{\Sigma}_n + \gamma_n \Sigma_{Y_n-\bar{Y}_n,Y_n-\bar{Y}_n} \gamma_n^* = \overline{\Sigma}_n. \tag{50}$$

We can substitute for γ_n, but let's use the transpose of (46) to save time. We obtain

$$\tilde{\Sigma}_n + \gamma_n H_n \overline{\Sigma}_n = \overline{\Sigma}_n. \tag{51}$$

Then substituting for γ_n we obtain

$$\tilde{\Sigma}_n + \overline{\Sigma}_n H_n^* \left(H_n \overline{\Sigma}_n H_n^* + \Sigma_{R_n,R_n}\right)^{-1} H_n \overline{\Sigma}_n = \overline{\Sigma}_n. \tag{52}$$

This gives the second equation of Problem 5.2.8. Thus equations (48) and (52) are the desired equations relating $\tilde{X}_n$ and $\tilde{\Sigma}_n$ to $\bar{X}_n$ and $\overline{\Sigma}_n$.