

Comments on the Markov property

1 Statements of the property

Here we consider alternate ways of writing the basic Markov property (Definition 2.2.1 in the text).

Definition 1 (Formulation in the text) *Let Z_0, Z_1, \dots be a sequence of \mathbb{S} -valued random variables. We say that Z_0, Z_1, \dots is a Markov chain in discrete time if, for arbitrary time $n \geq 0$ and $z_0, z_1, \dots, z_{n+1} \in \mathbb{S}$,*

$$P(Z_{n+1} = z_{n+1} | Z_0 = z_0, \dots, Z_n = z_n) = P(Z_{n+1} = z_{n+1} | Z_n = z_n), \quad (1)$$

whenever $P(Z_0 = z_0, \dots, Z_n = z_n) > 0$.

It is easy to see that an equivalent statement is

$$E[f(Z_{n+1}) | Z_0 = z_0, \dots, Z_n = z_n] = E[f(Z_{n+1}) | Z_n = z_n], \quad (2)$$

whenever $P(Z_0 = z_0, \dots, Z_n = z_n) > 0$, for every bounded function f on \mathbb{S} .

More generally, we claim that for every $m > n$ we have

$$E[f(Z_{n+1}, \dots, Z_m) | Z_0 = z_0, \dots, Z_n = z_n] = E[f(Z_{n+1}, \dots, Z_m) | Z_n = z_n], \quad (3)$$

whenever $P(Z_0 = z_0, \dots, Z_n = z_n) > 0$, for every bounded function f on \mathbb{S}^{m-n} . We will discuss shortly how to prove this.

Remark 1 (Adding the present) *Suppose for some m and n that (3) holds whenever $P(Z_0 = z_0, \dots, Z_n = z_n) > 0$, for every bounded function f on \mathbb{S}^{m-n} . Then*

$$E[f(Z_n, \dots, Z_m) | Z_0 = z_0, \dots, Z_n = z_n] = E[f(Z_n, \dots, Z_m) | Z_n = z_n], \quad (4)$$

whenever $P(Z_0 = z_0, \dots, Z_n = z_n) > 0$, for every bounded function f on \mathbb{S}^{m-n+1} .

This remark follows from Theorem 1.3.11 in the text.

2 Re-stating in terms of conditioning on a random variable

Using Problem 1.3.8 to justify the almost sure property, we can restate (2) as

$$E[f(Z_{n+1})|Z_0, \dots, Z_n] = E[f(Z_{n+1})|Z_n] \text{ a.s.}, \quad (5)$$

we can restate (3) as

$$E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_n] = E[f(Z_{n+1}, \dots, Z_m)|Z_n] \text{ a.s.}, \quad (6)$$

and we can restate Remark 1 in the case of conditioning on a random variable as:

Remark 2 (Adding the present) *Suppose for some m and n that (6) holds for every bounded function f on \mathbb{S}^{m-n} . Then*

$$E[f(Z_n, \dots, Z_m)|Z_0, \dots, Z_n] = E[f(Z_n, \dots, Z_m)|Z_n] \text{ a.s.} \quad (7)$$

for every bounded function f on \mathbb{S}^{m-n+1} .

3 Proof of (6)

Equalities in what follows are meant to hold almost surely.

The statement to be proved holds for $m = n + 1$ by (5). Assume that it holds for $m = n + k$ for some k . Let us now consider m such that $m = n + k + 1$.

Using Theorem 1.3.18 (Telescoping) we have

$$\begin{aligned} E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_n] &= \\ E[E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_{n+1}]|Z_0, \dots, Z_n] & \quad (8) \end{aligned}$$

Using Remark 2 and our inductive assumption we have

$$E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_{n+1}] = E[f(Z_{n+1}, \dots, Z_m)|Z_{n+1}] \quad (9)$$

That is, $E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_{n+1}]$ is actually a function of Z_{n+1} . Hence we then have

$$\begin{aligned} E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_n] &= \\ E[E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_{n+1}]|Z_n] & \quad (10) \end{aligned}$$

by the basic Markov property. Applying Theorem 1.3.18 (Telescoping) again, this gives

$$E[f(Z_{n+1}, \dots, Z_m)|Z_0, \dots, Z_n] = E[f(Z_{n+1}, \dots, Z_m)|Z_n]. \quad (11)$$

Thus we have proved our statement for $m = n + k + 1$, and hence, by induction, for all $m > n$.

This completes the proof of (6).