5654 Class comments 1, Spring 2015

1 Facts about Riemann-Stieltjes integration

The definition of a Riemann-Stieltjes sum is given in the notes in Section 1.2.3, on page 17. We repeat it here, but refer to the notes for the definition of the Riemann-Stieltjes integral.

Definition 1 (Riemann-Stieltjes sums for a partition) *Let f and g be arbitrary real-valued functions on an interval $[a, b]$ of $\mathbb{R}$. A partition P for $[a, b]$ is a finite sequence of of points $x_0, x_1, \ldots, x_n$ in $[a, b]$ such that $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$.*

A refinement Q of a partition P of $[a, b]$ is any partition of $[a, b]$ which contains P as a subsequence.

A Riemann-Stieltjes sum for the partition P and the functions f, g is any sum of the form

$$\sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1})),$$

where for each i, $\xi_i \in [x_{i-1}, x_i]$.

Definition 2 (The Cauchy property for Riemann-Stieltjes sums) *Let f and g be arbitrary real-valued functions on an interval $[a, b]$ of $\mathbb{R}$. Suppose that for each $\varepsilon > 0$ there exists a partition P_ε, such that for any partitions Q_1 and Q_2 which are refinements of P, any Riemann-Stieltjes sums s_1 for Q_1, f, g and any Riemann-Stieltjes sum s_2 for Q_2, f, g, $|s_1 - s_2| < \varepsilon$. Then we say that the Riemann-Stieltjes sums for f, g on $[a, b]$ have the Cauchy property.*

Lemma 1 (Alternate form of the Cauchy property) *Let f and g be arbitrary real-valued functions on an interval $[a, b]$ of $\mathbb{R}$. The following statements are eqivalent:*

(i) Tthe Riemann-Stieltjes sums for f, g on $[a, b]$ have the Cauchy property.

(ii) For each $\varepsilon > 0$ there exists a partition P_ε and a number L_ε, such that for any partitions Q which is a refinement of P, and any Riemann-Stieltjes sums s for Q, f, g, $|s - L_varepsilon| < \varepsilon$.

Proof

(i) $\implies$ (ii) Let $\varepsilon > 0$ be given. By the Cauchy property there exists a partition P_ε, such that for any partitions Q_1 and Q_2 which are refinements of P, any Riemann-Stieltjes sums s_1 for Q_1, f, g and any Riemann-Stieltjes sum s_2 for Q_2, f, g, $|s_1 - s_2| < \varepsilon$.

Let L_ε be any Riemann-Stieljes sum for P_ε. For any partition Q which is a refinement of P_ε, and any Riemenn-Stieltjes sum s for Q, f, G, taking $Q_1 = Q$, $Q_2 = P_\varepsilon$, $s_1 = s$ and $s_2 = L_\varepsilon$ in the previous statement shows that (ii) holds.

(ii) $\implies$ (i) Let $\varepsilon > 0$ be given. By (ii), there exists a partition P_ε and a number L_ε, such that for any partitions Q which is a refinement of P, and any Riemann-Stieltjes sums s for Q, f, g, $|s - L| < \varepsilon/2$. Let Q_1 and Q_2 be partitions which are refinements of P, let s_1 be any Riemann-Stieltjes sum for Q_1, f, g and let s_2 be any Riemann-Stieltjes sum for Q_2, f, g,

By (ii), $|s_1 - L_\varepsilon| < \varepsilon/2$ and $|s_2 - L_\varepsilon)\varepsilon| < \varepsilon/2$. Hence $|s_1 - s_2| < \varepsilon$, so the Cauchy property holds.

$\square$

Lemma 2 (The Cauchy criterion for Riemann-Stieltjes sums) *Let f and g be real-valued functions on an interval $[a, b]$ of $\mathbb{R}$. Suppose that the Riemann-Stieltjes sums for f, g on $[a, b]$ have the Cauchy property. Then the Riemann-Stieltjes integral $\int_a^b f\, dg$ exists.*

Proof The proof is similar to the proof of the corresponding fact for sequences, and is omitted.

Lemma 3 (Additivity on pieces) *Let f and g be arbitrary real-valued functions on an interval $[a, b]$ of $\mathbb{R}$, and let $c \in [a, b]$. Suppose that $\int_a^c f\, dg$ exists and $\int_c^b f\, dg$ exists. Then $\int_a^b f\, dg$ exists and*

$$\int_a^b f\, dg = \int_a^c f\, dg + \int_c^b f\, dg.$$

Proof Let $\varepsilon > 0$ be given.

Let P_ε^1 be a partition for $[a, c]$ such that for partition Q_1 of $[a, c]$ which is a refinment of P_ε^1, and any Riemann-Stieltjes sum s_1 for Q_1,

$$\left| \int_a^c f\, dg - s_1 \right| < \varepsilon.$$

Let P_ε^2 be a partition for $[c, b]$ such that for partition Q_2 of $[c, b]$ which is a refinment of P_ε^2, and any Riemann-Stieltjes sum s_2 for Q_2,

$$\left| \int_c^b f\, dg - s_2 \right| < \varepsilon.$$

Let P_ε be the sequence of points obtained by concatenating the sequences P_ε^1 and P_ε^2 in an appropriate manner. More precisely, if P_ε^1 is the sequence $x_0, x_1, \ldots, x_n$ and P_ε^2 is the sequence $y_0, y_1, \ldots, y_n$, let P_ε be the sequence

$$x_0, x_1, \ldots, x_n, y_1, y_2, \ldots, y_m.$$

Denote the sequence P_ε by $z_0, \ldots, z_{m+n}$. Since $x_0 = a$ we have $z_0 = a$. Since $y_n = b$ we have $z_{m+n} = b$. Since $x_n = c$ we have $z_n = c$. Since also $y_0 = c = z_n$, we have $y_j = z_{n+j}$ for $j = 0, 1, \ldots, m$.

Let Q a partition of $[a, b]$ which is a refinement of P_ε. Let Q be the sequence of points $t_0, \ldots, t_k$, where $a = t_0$ and $b = t_k$. Since Q is a refinement of P_ε, there is some index $r \in \{0, \ldots, k\}$ such that $t_r = c$. Let Q_1 be the sequence $t_0, \ldots, t_r$ and let Q_2 be the sequence $t_r, \ldots, t_k$. Then Q_1 is a partition of $[a, c]$ which is a refinement of P_ε^1 and Q_2 is a partition of $[c, b]$ which is a refinement of P_ε^2.

For any Riemann-Stieljes sum s for Q, f, G on $[a, b]$, let

$$s = \sum_{i=1}^{k} f(\xi_i)\left(g(t_i) - g(t_{i-1})\right),$$

where ξ_i be a point in $[t_{i-1}, t_i]$ for each $i = 1, \ldots, k$. Let

$$s_1 = \sum_{i=1}^{r} f(\xi_i)\left(g(t_i) - g(t_{i-1})\right), \; s_2 = \sum_{i=r+1}^{k} f(\xi_i)\left(g(t_i) - g(t_{i-1})\right).$$

Then Let f and g be arbitrary real-valued functions on an interval $[a, b]$ s_1 is a Riemann-Stieltjes sum for Q_1, f, g on $[a, c]$, s_2 is a Riemann-Stieltjes sum for Q_2, f, g obn $[c, b]$.

It is easy to check that Q_1 is a refinement of P_ε^1 and Q_2 is a refinement of P_ε^2. Hence

$$\left|\int_a^c f\,dg - s_1\right| < \varepsilon, \; \left|\int_a^c f\,dg - s_1\right| < \varepsilon,$$

and so

$$\left|\int_a^c f\,dg + \int_c^b f\,dg - s\right| < 2\varepsilon.$$

The lemma follows.

□

We have $\int_a^b f\,dg = \int_a^b f(x)\,dx$ when $g(x) = x$ for all x. Thus a special case of the next lemma shows that any continuous function on a closed finite interval is Riemann-integrable.

Lemma 4 (A basic case) *Let f be a continuous real-valued function on $[a, b]$ and let g be monotonic increasing real-valued function $[a, b]$. Then $\int_a^b f\,dg$ exists.*

Proof Since f is continuous on the closed bounded interval $[a, b]$, it is uniformly continuous on this interval.

Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that for any $x, y \in [a, b]$ with $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$.

Let P_ε be any partition $a = x_0 \leq x_1 \leq \ldots \leq x_n = b$ such that $x_i - x_{i-1} < \delta$ for $i = 1, \ldots, n$. Let

$$L_\varepsilon = \sum_{i=1}^{n} f(x_i)(g(x_i) - g(x_{i-1})).$$

Let Q be any partition of $[a, b]$ which is a refinement of P_ε. Let the partition points be $a = y_0 \leq y_1 \leq \ldots \leq y_m = b$, and let

$$s = \sum_{j=1}^{m} f(\eta_j)(g(y_j) - g(y_{j-1})) \tag{1}$$

be a Riemann-Stieltjes sum for Q, f, g on $[a, b]$, where $\eta_j \in [y_{j-1}, y_j]$ for each j.

Since $x_0, \ldots, x_n$ is a subsequence of $y_0, \ldots, y_m$, for each $i = 0, 1, \ldots, n$ there exists an inteqer r_i such that $x_i = y_{r_i}$. We can choose the integers r_i such that $0 = r_0 \leq r_1 \leq r_n = n$. Then

$$s = \sum_{i=1}^{n} \sum_{j=q_{i-1}+1}^{q_i} f(\eta_j)(g(y_j) - g(y_{j-1})).$$

Let $\eta'_j = x_i$ for $q_{i-1} + 1 \leq j \leq q_j$. Let

$$s' = \sum_{j=1}^{m} f\left(\eta'_j\right)(g(y_j) - g(y_{j-1})), \tag{2}$$

so that

$$\begin{aligned} s' = \sum_{i=1}^{n} \sum_{j=q_{i-1}+1}^{q_i} f\left(\eta'_j\right)(g(y_j) - g(y_{j-1})) &= \sum_{i=1}^{n} \sum_{j=q_{i-1}+1}^{q_i} f(x_i)(g(y_j) - g(y_{j-1})) \\ &= \sum_{i=1}^{n} f(x_i)(g(x_i) - g(x_{i-1})) = L_\varepsilon. \end{aligned}$$

For $q_{i-1} + 1 \leq j \leq q_i$, we have $[y_{q_{i-1}}, y_{q_i}] \subset [x_{i-1}, x_i]$. Hence $\left|\eta'_j - \eta_j\right| < \delta$, and so $\left|f\left(\eta'_j\right) - f(\eta_j)\right| < \varepsilon$. Comparing (1) and (2), we have

$$|s - L_\varepsilon| \leq \sum_{i=1}^{n} \sum_{j=q_{i-1}+1}^{q_i} \varepsilon(g(y_j) - g(y_{j-1})) = \varepsilon(g(b) - g(a)).$$

Thus the Riemann-Stieltjes partitions for f, g on $[a, b]$ have the Cauchy property, and so $\int_a^b f\, dg$ exists.

□