5654 Homework 1 solutions

Problem 1

Let X be a nonnegative random variable with finite range. Give a complete proof of the formula

$$EX = \int_0^\infty (1 - F_X(t))\, dt$$

which is given in 1.2.2. In writing down your proof, denote the elements of the range of X by $a_1, \ldots, a_k$, where $a_1 < \ldots < a_k$.

Solution There are lots of ways to write up a proof. Here is one.

By the definition of expected value of a discrete random variable,

$$EX = \sum_{i=1}^{n} a_i P(X = a_i).$$

Let $a_0 = 0$. Then

$$EX = \sum_{i=1}^{k} \sum_{j=1}^{i} (a_j - a_{j-1}) P(X = a_i) = \sum_{j=1}^{k} (a_j - a_{j-1}) \sum_{i=j}^{k} P(X = a_i)$$
$$= \sum_{j=1}^{n} (a_j - a_{j-1}) P(X > a_{j-1}) = \sum_{j=1}^{k} (a_j - a_{j-1}) (1 - F_X(a_{j-1})).$$

Since $F_X(t) = F_X(a_{j-1})$ on the interval $[a_{j-1}, a_j)$, it is easy to check from the definitions that the final sum is $\int_0^{a_k} (1 - F_X(t))\, dt$. Since $1 - F_X(t) = 0$ for $t \geq a_k$, this integral is the same as $\int_0^\infty (1 - F_X(t))\, dt$.

Problem 1.2.9

Assume the integral exists. We will prove a contradiction exists.

Let $\varepsilon = p^2/2$. Let P_ε be a partition of $[0, 5]$ with the property stated in the definition of Riemann-Stieltjes integrability on page 17, so that for every refinement Q of P_ε, any Riemann-Stieltjes sums using Q, g, g are within ε of $\int_0^5 g\, dg$.

Let Q be any refinement of P_ε which includes the point 2.

Then Q is a partition using points $0 = y_0 \leq y_1 \leq \ldots \leq y_n$ such that $y_j = 2$ for some index j.

Let k be the last index before j such that $y_k < y_j$. Then $y_{k+1} = y_j = 2$. Let s_1 be a Riemann-Stieltjes sum for the partition Q using evaluation points ξ_i for each interval $[y_{i-1}, y_i]$, such that $\xi_{k+1} = y_k < 2$, so that $g(\xi_{k+1}) = 0$.

Let s_2 be a Riemann-Stieltjes sum for the partition Q using evaluation points $\tilde{\xi}_i$, such that $\tilde{\xi}_i = \xi_i$ for each interval $[y_{i-1}, y_i]$ *except* that $\tilde{\xi}_{k+1} = 2$, so that $g\left(\tilde{\xi}_{k+1}\right) = p$.

Since $g(y_{k+1}) - g(y_k) = g(2) - g(y_k) = p - 0 = p$, we see that $s_2 - s_1 = p^2$. But by assumption, $\left|s_i - \int_0^5 g\, dg\right| < \varepsilon = p^2/2$, so $|s_1 - s_2| < p^2$, contradiction.

Problem 1.2.26

We are given that G, H, K are independent. According to the definition at the top of page 27, this means that the indicators I_G, I_H, I_K form an independent family of random variables. Hence in particular

$$P(I_G = 1, I_H = 0, I_K = 1) = P(I_G = 1)\, P(I_H = 0)\, P(I_K = 1).$$

This is exactly the statement that

$$P(GHK) = P(G)\, P(H)\, P(K).$$

Problem 1.3.2

There are several ways to write down a justification for each formula in this problem. Because of the text's convention about conditioning on events of probability zero, we will have to be careful to include that case in our work.

As noted in class, and in the text on page 32, we always have the multiplied-through form of the conditional probability formula,

$$P(GH) = P(G|H)\, P(H),$$

even in the case that $P(H) = 0$. So we can use that freely.

Justification of (i) Using the multiplied-through form of the conditional probability formula twice, we have

$$P(GHF) = P(G|HF)\, P(HF) = P(G|HF)\, P(H|F)\, P(F).$$

Using the multiplied-through form of the conditional probability formula once, we have

$$P(GHF) = P(GH|F)\, P(F).$$

Thus

$$P(G|HF)\, P(H|F)\, P(F) = P(GH|F)\, P(F).$$

If $P(F) > 0$ we can divide both sides of this equation by $P(F)$, and obtain equation (i). On the other hand, if $P(F) = 0$ the text's definition of conditional probability says that $P(H|F) = 0$ and $P(GH|F) = 0$, so (i) just says $0 = 0$, which is again true. Thus (i) holds in all cases.

Justification of (ii) Using the multiplied-through form of the conditional probability formula,

$$P(GH) = P(G|H)\,P(H)\,.$$

Hence if $P(G|H) = 0$ we obviously have $P(GH) = 0$.

Conversely, if $P(GH) = 0$ then $P(G|H)\,P(H) = 0$. This last equation forces $P(G|H) = 0$ if $P(H) > 0$. On the other hand if $P(H) = 0$ then $P(G|H) = 0$ by the text's definition. So (ii) holds.

Justification of (iii) If $P(HK) = 0$ then $P(G|HK) = 0$ by the text's definition and $P(GH|HK) = 0$ by the text's definition. So (iii) holds in this case.

Now we assume $P(HK) > 0$. Then

$$P(GH|HK) = \frac{P(GHHK)}{P(HK)} = \frac{P(GHK)}{P(HK)} = P(G|HK)\,,$$

so (iii) holds in this case also. Incidentally, notice that H is certain to happen given HK, so (iii) is physically clear.

Problem 1.3.4

(i) If $P(HK) > 0$, then $P(H) > 0$, so we can use the ordinary definition for conditional expectation.

We have

$$\begin{aligned} P(GK|H) &= \frac{P(GHK)}{P(HK)} = \frac{P(GH)\,P(K)}{P(H)\,P(K)} \\ &= \frac{P(GH)}{P(H)} = P(G|H)\,. \end{aligned}$$

We are also asked to show that $P(G|HK) = P(G|H)$ if $P(K) = 1$. Note that if $P(K) = 1$ then from the definition we see that K is independent of *any* set. Hence the previous fact can be applied if $P(HK) > 0$. If $P(HK) = 0$, then since $H = (HK) \cup (HK^c)$ we have $P(H) = P(HK) + P(K^c) = 0$. Hence by our convention we have $P(G|HK) = 0 = P(G|H)$

(ii) If $P(H) = 0$ then the stated equation holds by definition. Assume from now on that $P(H) > 0$.

We have $P(GKH) = P(GH)\,P(K)$. Hence

$$\frac{P(GKH)}{P(H)} = \left(\frac{P(GH)}{P(H)}\right) P(K)\,,$$

so $P(GK|H) = P(G|H)\,P(K)$ as claimed.