Solution Set 1

Topic 1

0001-1:

Multiply the rational function by $\frac{x^{-5}}{x^{-5}}$, -5 chosen because it is the highest power in the denominator. Doing so results in

$$\lim_{x\to\infty}\frac{5x-\frac{4}{x}+\frac{2}{x^4}-\frac{8}{x^5}}{3+\frac{1}{x^3}-\frac{7}{x^4}-\frac{1}{x^5}}$$

Thus, all terms with x in the denominator go to 0 in the limit, and $\frac{5x}{3}$ goes to infinity in the limit.

Note: Several applications of L'Hopital's Rule will give the same result.

0001-2: When plugging in 0, the result is the indeterminate form $\frac{0}{0}$. This means we must use L'hopital's Rule, and after applying it, we discover we arrive at another indeterminate form. After several applications of L'Hopital's Rule, we arrive at the result $\frac{7}{24}$.

0001-3: Set

$$\lim_{x\to\infty}(1-\frac{.08}{n})^n = y$$

and take natural log of both sides to get

$$\lim_{x\to\infty} n\ln(1-\frac{.08}{n}) = \ln y$$

which is the same as

$$\lim_{x\to\infty}\frac{\ln(1-\frac{.08}{n})}{\frac{1}{n}} = \ln y$$

which is the interminate $\frac{0}{0}$ on the left side of the equation, so we apply L'Hopital's Rule to get

$$\lim_{x\to\infty}\frac{\frac{1}{1-.08/n}\frac{.08}{n^2}}{\frac{-1}{n^2}} = \ln y$$

or

$$\lim_{x\to\infty}\frac{-.08}{1-.08/n} = \ln y$$

or

$$-.08 = \ln y$$

i.e.

$$y = e^{-.08}$$

0001-4: Here you want to use the exact same method as above. The extra term winds up going away, and we have the same result, i.e.

$$\lim_{x\to\infty}\left(1-\frac{.08}{n}+\frac{.14}{n^2}\right)^n = e^{-.08}$$

0001-5: By using the same method as problem 1, i.e., by dividing the numerator and denominator by x^{250}, we conclude that the limit is $\frac{4}{7}$.

0001-6: This can be rewritten slightly as

$$\lim_{x\to\infty} \frac{1}{x^{2x^2-2x+1}}$$

and we see that the denominator increases without bound as $x \to \infty$. Thus the limit is equal to 0.

0001-7: Replace e^{4x} by its power series

$$e^{4x} = 1 + (4x) + \frac{(4x)^2}{2!} + \frac{(4x)^3}{3!} + ...$$

and $\sin x$ by its power series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

to write

$$f(x) = 1 - 8x^2 + \text{higher powers of } x$$

We now appeal to problem 4 and see the resulting limit is equal to e^{-8x^2}.

0001-8: Here, replace x by $x+4$ to obtain

$$\begin{aligned}\int_{-\infty}^{\infty} e^{\frac{-(x+4)^2}{2}+4(x+4)+8}dx &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-4x-8+4x+16+8}dx \\ &= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}+16}dx \\ &= \sqrt{2\pi}e^{16}\end{aligned}$$

0001-9: First note that

$$(25x-75)_+ = \begin{cases} 25x-75 & \text{if} \quad x \geq 3 \\ 0 & \text{if} \quad x < 3 \end{cases}$$

Thus,

$$\int_{-\infty}^{\infty} (25x-75)_+ e^{-x^2/2}dx = \int_{3}^{\infty} (25x-75)e^{-x^2/2}dx$$

We can split this result into two integrals, the first solvable by substitution letting $u = x^2/2$

$$\int_{3}^{\infty} 25xe^{-x^2/2} = \int_{9/2}^{\infty} 25e^{-u} = 25e^{-9/2}$$

The second integral is equal to $-75\sqrt{2\pi}[\Phi - 3)]$, and so the final solution is

$$25e^{-9/2} - 75\sqrt{2\pi}[\Phi(-3)]$$

0001-10: Without loss of generality, we can define $X : [0,1] \to \mathbb{R}$ by

$$X(\omega) = \begin{cases} 2 & \text{if} \quad 0 \leq \omega \leq .75 \\ 9 & \text{if} \quad .75 < \omega \leq 1 \end{cases}$$

Thus, the mean is

$$\begin{aligned} E(X) &= \int_0^1 X(\omega)d\omega \\ &= 2(.75) + 9(.25) \\ &= 3.75 \end{aligned}$$

The standard deviation is the square root of the variance, and variance is

$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= (4(.75) + 81(.25)) - (3.75)^2 \\ &= 9.1875 \end{aligned}$$

Thus the standard deviation is

$$SD[X] = \sqrt{Var[X]} = 3.031$$

0001-11: Without loss of generality, we can define $X : [0, 1] \to \mathbb{R}$ by

$$X(\omega) = \begin{cases} 1 & \text{if} \quad 0 \le \omega \le .75 \\ 8 & \text{if} \quad .75 < \omega \le 1 \end{cases}$$

Thus

$$E(X) = 1(.75) + 8(.25) = 2.75$$

and

$$Var[X] = 1(.75) + 64(.25) - (2.75^2) = 9.1875$$

and so the standard deviation is again 3.031.

0001-12: First, let us calculate:

$$E[W] = -2.5; \qquad E[X] = 7.1; \qquad E[Y] = 7000300 = 7 \times 10^6; \qquad E[Z] = 1.6999 = 1.70$$

$$Var[W] = .25; \qquad Var[X] = 1.89; \qquad Var[Y] = 2.10 \times 10^{13}; \qquad Var[Z] = .21$$

$$SD[W] = .5; \qquad SD[X] = 1.37; \qquad SD[Y] = 4.58 \times 10^6; \qquad SD[Z] = .46$$

Furthermore,

$$W + X = \begin{cases} 3 & \text{if} \quad 0 \le \omega < .3 \\ 6 & \text{if} \quad .3 \le \omega < .5 \\ 5 & \text{if} \quad .5 \le \omega \le 1 \end{cases} \quad ; \qquad \begin{cases} E[W+X] & = 4.6 \\ Var[W+X] & = 6.64 \end{cases}$$

$$W + Y = \begin{cases} 10^3 - 2 & \text{if} \quad 0 \le \omega < .3 \\ 10^7 - 2 & \text{if} \quad .3 \le \omega < .5 \\ 10^7 - 3 & \text{if} \quad .5 \le \omega \le 1 \end{cases} \quad ; \qquad \begin{cases} E[W+Y] & = 7 \times 10^6 \\ Var[W+Y] & = 2.1 \times 10^{13} \end{cases}$$

$$X + Y = \begin{cases} 10^3 + 5 & \text{if} \quad 0 \le \omega < .3 \\ 10^3 + 8 & \text{if} \quad \omega = .3 \\ 10^7 + 8 & \text{if} \quad .3 < \omega \le 1 \end{cases} \quad ; \qquad \begin{cases} E[X+Y] & = 7 \times 10^6 \\ Var[X+Y] & = 2.1 \times 10^{13} \end{cases}$$

$$X + Z = \begin{cases} 6 & \text{if} \quad 0 \le \omega < .3 \\ 9 & \text{if} \quad .3 \le \omega \le .3001 \\ 10 & \text{if} \quad .3001 < \omega \le 1 \end{cases} \quad ; \qquad \begin{cases} E[X+Z] & = 8.80 \\ Var[X+Z] & = 3.36 \end{cases}$$

$$Y + Z = \begin{cases} 10^3 + 1 & \text{if} \quad 0 \leq \omega \leq .3 \\ 10^7 + 1 & \text{if} \quad .3 < \omega \leq .3001 \\ 10^7 + 2 & \text{if} \quad .3001 < \omega \leq 1 \end{cases} \quad ; \qquad \begin{cases} E[Y+Z] & = 7 \times 10^6 \\ Var[Y+Z] & = 2.1 \times 10^1 3 \end{cases}$$

Now using the formula

$$Cov[A, B] = \frac{Var[A+B] - Var[A] - Var[B]}{2}$$

we calculate

$$Cov[W, X] = 2.25; \quad Cov[W, Y] = -1.5 \times 10^6; \quad Cov[X, Y] = 6.30 \times 10^6;$$

$$Cov[X, Z] = .63; \quad Cov[Y, Z] = 2.10 \times 10^6$$

Note that the variances used to calculate the Covariance are exact, but the variances depicted above are rounded. Finally, we can calculate the Correlation using the formula

$$Corr[A, B] = \frac{Cov[A, B]}{SD[A]SD[B]}$$

and thus we arrive at

$$\begin{aligned} Corr[W, X] &= -.65 \\ Corr[W, Y] &= -.65 \\ Corr[X, Y] &= 1.00 \\ Corr[X, Z] &= 1.00 \\ Corr[Y, Z] &= 1.00 \end{aligned}$$

Topic 2

0002-1: Gail hedges this contract by buying 100 shares of Borogrove stock at $5.00 per share, and then taking out $x/1.022 in a risk-free bank loan. Thus the current value of this contract is $500 - x/1.022$, and by setting that value to 0, we find that $x = 511$.

0002-2: First we must calculate the risk-neutral probability of an uptick and a downtick. Thus we want to calculate

$$1.04p + .94(1-p) = 1.03$$

where p is the probability of an uptick and $1 - p$ is the probability of a downtick. We thus find that $p = .9$, or 90%. bf a) To price a contract that pays $1 on the uptick and $0 on the downtick, we calculate the value of the expected payoff i.e.

$$\frac{1 \times .9 + 0 \times .1}{1.03} = .8738$$

b) Similarly, for a contract that pays $0 on the uptick and $1 on the downtick, we calculate

$$\frac{0 \times .9 + 1 \times .1}{1.03} = .0971$$

c) Lastly, for a contract that pays \$1 on the uptick and \$1 on the downtick, we calculate

$$\frac{1 \times .9 + 1 \times .1}{1.03} = .9709$$

We see that

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

0002-3: Let x be the quantity of asset, say stock in Borogrove, and y be the amount of the bank loan. Then

$$x \times \left\{ \$1 \begin{array}{l} \nearrow \ \$1.04 \\ \searrow \ \$0.94 \end{array} \right. \quad \text{and} \quad -y \times \left\{ \$1 \begin{array}{l} \nearrow \ \$1.03 \\ \searrow \ \$1.03 \end{array} \right.$$

a) From **0002-2a)**, we know the contract price is \$.8738, so

$$.8738 \begin{array}{l} \nearrow \ \$1 \\ \searrow \ \$0 \end{array}$$

Thus,

$$x = \frac{1 - 0}{1.04 - .94} = 10 \text{ shares}$$

The dollar value of 10 shares is \$10, so we know the price of the bank loan is

$$10 - .8738 = 9.1262$$

Hence, our hedging portfolio is

$$\begin{cases} 10 \text{ shares} & \text{Purchase 10 shares} \\ -\$9.13 & \text{Take out a bank loan of \$9.13} \end{cases}$$

b) Again, from **0002-2b)**, we know the contract price is \$0.0971, so

$$.0971 \begin{array}{l} \nearrow \ \$0 \\ \searrow \ \$1 \end{array}$$

Thus,

$$x = \frac{0 - 1}{1.04 - .94} = -10 \text{ shares}$$

Since the value of -10 shares is $-\$10$, the bank loan is

$$-10 - .0971 = -10.0971$$

which actually means a deposit of \$10.0971. Thus, our hedging portfolio is

$$\begin{cases} \text{-10 shares} & \text{Short 10 shares} \\ \$10.0971 & \text{Deposit \$10.0971 into the bank} \end{cases}$$

c) Here,

$$x = \frac{1 - 1}{1.04 - .94} = 0 \text{ shares}$$

From **0002-2c)**, we know the contract price is \$0.9709, and the value of 0 shares is \$0, so the bank loan is

$$0 - .9709 = -\$.9709$$

which means a deposit of \$0.9709. Thus, our hedging portfolio is

$$\begin{cases} 0 shares \\ -\$0.9709 & \text{Deposit \$.9709 into the bank} \end{cases}$$

d)

$$\text{Shares in } \mathbf{a)} + \text{Shares in } \mathbf{b)} = \text{Shares in } \mathbf{c)}$$

and

$$\text{loan in } \mathbf{a)} + \text{loan in } \mathbf{b)} = \text{loan in } \mathbf{c)}$$

Topic 3

0003-1

a) We know that that

$$\frac{\mu}{N} = E[\ln S] = Pr[\ln S = u]u + Pr[\ln S = d]d$$

where $Pr[\ln S = u] = .55$ and $Pr[\ln S = d] = .45$. We also know that

$$\frac{\sigma}{\sqrt{N}} = SD[\ln S] = \sqrt{(.55)(.45)}(u - d)$$

Solving these two equations for u and d gives

$$u = \frac{\mu}{N} + \sqrt{\frac{9}{11N}}\sigma = \frac{\mu}{N} + .9045\frac{\sigma}{\sqrt{N}}$$

and

$$d = \frac{\mu}{N} - \sqrt{\frac{11}{9N}}\sigma = \frac{\mu}{N} - 1.1055\frac{\sigma}{\sqrt{N}}$$

b) Let p be the probabiltiy of the risk-neutral uptick and $q = 1 - p$ be the probability of the risk-neutral downtick. Then since e^r is the amount a dollar grows to after time T, $e^{r/N}$ is the amount a dollar will grow after one subinterval. Thus,

$$pe^u + qe^d = e^{r/N} = pe^u + (1 - p)e^d$$

Solving for p, we get

$$p = \frac{e^{r/N} - e^d}{e^u - e^d}$$

where

$$u = \frac{\mu}{N} + \sqrt{\frac{9}{11N}}\sigma \qquad \text{and} \qquad d = \frac{\mu}{N} - \sqrt{\frac{11}{9N}}\sigma$$

from part **a)**. Likewise, solving for q in a similar manner gives

$$q = \frac{e^{r/N} - e^u}{e^d - e^u}$$

with the same values of u and d above.