0024-1: a) Let $Y := W_3/\sqrt{3}$ and $Z := W_4 - W_3$. Since W_t is a Brownian motion, Y and Z are standard normal and iid. Thus,

$$\begin{aligned} E[W_3^2 W_4^3] &= E[3Y^2(\sqrt{3}Y + Z)^3] \\ &= E[3Y^2(3\sqrt{3}Y^3 + 9Y^2 Z + 3\sqrt{3}YZ^2 + Z^3)] \\ &= E[9\sqrt{3}Y^5 + 27Y^4 Z + 9\sqrt{3}Y^3 Z^2 + 3Y^2 Z^3] \\ &= 9\sqrt{3}E[Y^5] + 27E[Y^4]E[Z] + 9\sqrt{3}E[Y^3]E[Z^2] + 3E[Y^2]E[Z^3] \\ &= 0 \end{aligned}$$

because $E[Y^k] = E[Z^k] = 0$ for odd k.

b) We know that

$$E\left[\int_0^5 W_t^3 dt + (e^{W_t} - e)_+ dW_t\right] = E\left[\int_0^5 W_t^3 dt\right] + E\left[\int_0^5 (e^{W_t} - e)_+ dW_t\right]$$

Noting that $E[W_t^3] = 0$, we see that

$$E\left[\int_0^5 W_t^3 dt\right] = \int_0^5 E[W_t^3] dt = 0$$

Now let $\mathcal{F}_\bullet$ be the filtration of the Brownian motion W_t Then

$$E\left[\int_0^5 (e^{W_t} - e)_+ dW_t\right] = E\left[E\left[\int_0^5 (e^{W_t} - e)_+ dW_t | \mathcal{F}_0\right]\right]$$

by the Power Tower Law.

Now we see that

$$\begin{aligned} E\left[\int_0^5 (e^{W_t} - e)_+ dt\right] &= \int_0^5 (E(e^{W_t} - e)_+ dt \\ &= \int_)^5 (e^{t/2} - e)_+ dt \\ &= \int_2^5 e^{t/2} - e dt \\ &= 2e^{5/2} - 5e \\ &< \infty \end{aligned}$$

Since $U_5 = \int_0^5 (e^{W_t} - e)_+ dW_t$, and since we've just shown that $H_t := (e^{W_t} - e)_+$ is $\mathcal{F}_\bullet$-adapted and L^2, $U_\bullet$ is an $\mathcal{F}_\bullet$-martingale, and

$$E[U_5|\mathcal{F}_0] = U_0 = 0$$

Thus,

$$E\left[E\left[\int_0^5 (e^{W_t} - e)_+ dW_t | \mathcal{F}_0\right]\right] = E[0] = 0]$$

Putting it all together, we see that

$$E\left[\int_0^5 W_t^3 dt + (e^{W_t} - e)_+ dW_t\right] = 0$$

c) With similar reasoning to that which was used in part **b)**, we can show that

$$E\left[\int_0^5 e^{4W_t - 2} dW_t\right] = 0$$

Now

$$\begin{aligned} E\left[\int_0^5 W_t^4 dt\right] &= \int_0^5 E[W_t^4] dt \\ &= \int_0^5 3t^2 dt \\ &= 125 \end{aligned}$$

Thus,

$$E\left[\int_0^5 W_t^4 dt + e^{4W_t - 2} dW_t\right] = 125$$

Topic 25

0025-1: The maximal flowline $c(t)$ for V footed at $(-1, 2)$ is given by the solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 1 \\ \frac{dy}{dt} &= 3 \end{aligned}$$

with initial condition $(x, y)_{t=0} = (-1, 2)$. Hence, the solution is

$$\begin{aligned} x(t) &= t - 1 \\ y(t) &= 3t + 2 \end{aligned}$$

or

$$y = 3x + 5$$

0025-2: The maximal flowline $c(t)$ for V footed at $(-1, 1)$ is given by the solution of the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= 2x \\ \frac{dy}{dt} &= -y \end{aligned}$$

with initial condition $(x, y)_{t=0} = (-1, 1)$. Hence, the solution is

$$\begin{aligned} x(t) &= -e^2 t \\ y(t) &= e^- t \end{aligned}$$

or

$$y = \frac{1}{\sqrt{-x}}$$

0025-3: The characteristic equation of M is

$$\lambda^2 + 2\lambda - 8$$

which gives eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -4$. This gives possible eigenvectors $v_2 = (7, 2)^\top$ and $v_{-4} = (11, 3)^\top$. Define C^{-1} by

$$C^{-1} = \begin{bmatrix} 7 & 11 \\ 2 & 3 \end{bmatrix}$$

and thus

$$C = \begin{bmatrix} -3 & 11 \\ 2 & -7 \end{bmatrix}$$

Thus,

$$CMC^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

0025-4: The maximal flowline $c(t)$ for V footed at $(1, 1)$ is given by the solution of the system of differential equations

$$\frac{dx}{dt} = -130x + 462y$$
$$\frac{dy}{dt} = -36x + 128y$$

or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -130 & 462 \\ -36 & 128 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}$$

with initial condition $(x, y)_{t=0} = (1, 1)$. Thus, the solution to this system is

$$\begin{aligned}
\begin{bmatrix} x \\ y \end{bmatrix} &= e^{Mt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\text{(from problem \textbf{0025-3}, we substitute)} \quad &= e^{C^{-1}DCt} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= C^{-1} e^{Dt} C \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 7 & 11 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-4t} \end{bmatrix} \begin{bmatrix} -3 & 11 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -21e^{2t} + 22e^{-4t} & 77e^{2t} - 77e^{-4t} \\ -6e^{2t} + 6e^{-4t} & 22e^{2t} - 21e^{-4t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 56e^{2t} - 55e^{-4t} \\ 16e^{2t} - 15e^{-4t} \end{bmatrix}
\end{aligned}$$

0025-5: If X_t is a solution to

$$dX_t = 2dW_t + 4dt$$

then

$$X_t = 2W_t + 4t$$

Now $Var[X_7 - X_2] = 5$, so $Var[(X_7 - X_2)/\sqrt{5}] = 1$ and $Var[W_2] = 2$, so $Var[W_2/\sqrt{2}] = 1$. Define $Y := W_2/\sqrt{2}$ and $Z := (X_7 - X_2)/\sqrt{5}$. Then Y and Z are standard normal and iid and hence,

$$\begin{aligned}
E[e^{X_2+X_7}] &= E[e^{2W_2+8+2W_7+28}] \\
&= E[e^{2\sqrt{2}Y+2(\sqrt{5}Z+\sqrt{2}Y)+36}] \\
&= e^{36}E[e^{4\sqrt{2}Y+2\sqrt{5}Z}] \\
&= e^{36}E[e^{4\sqrt{2}Y}] \cdot E[e^{2\sqrt{5}Z}] \\
&= e^{36}e^{16}e^{10} \\
&= e^{62}
\end{aligned}$$