Topic 26

0026-1: a) $W_\bullet$ is Brownian motion, and $X_t = \arctan(W_t)$ and so

$$dX_t = \frac{1}{1+W_t^2}dW_t - \frac{W_t}{(1+W_t^2)^2}dt$$

b) $Yt = e^{W_t}$ and so

$$dY_t = e^{W_t}dW_t + \frac{e^{W_t}}{2}dt$$

c) $X_tY_t = \arctan(W_t)e^{W_t}$ and so

$$\begin{aligned} d(X_tY_t) = &\left[\frac{e^{W_t}}{1+W_t^2} + e^{W_t}\arctan(W_t)\right]dW_t \\ &+ \frac{1}{2}\left[\frac{e^{W_t}(1+W_t^2) - 2W_te^{W_t}}{(1+W_t^2)^2} + \frac{e^{W_t}}{1+W_t^2} + e^{W_t}\arctan(W_t)\right]dt \end{aligned}$$

d) We will employ the identities $(dW_t)^2 = dt, (dW_t)(dt) = 0$, and $(dt)^2 = 0$. Thus from **a), b) and c)**, we get

$$[(X_t)(dY_t)] = \arctan(W_t)\left(e^{W_t}dW_t + \frac{e^{W_t}}{2}dt\right) \tag{1}$$

$$[(dX_t)(Y_t)] = \left(\frac{1}{1+W_t^2}dW_t - \frac{W_t}{(1+W_t^2)^2}dt\right)e^{W_t} \tag{2}$$

$$[(dX_t)(dY_t)] = \left(\frac{1}{1+W_t^2}dW_t - \frac{W_t}{(1+W_t^2)^2}dt\right)\left(e^{W_t}dW_t + \frac{e^{W_t}}{2}dt\right) \tag{3}$$

$$= \frac{e^{W_t}}{1+W_t^2}dt \tag{4}$$

Thus,

$$\begin{aligned} [(X_t)(dY_t)] + [(dX_t)(Y_t)] + [(dX_t)(dY_t)] = &\left(\arctan(W_t)e^{W_t} + \frac{e^{W_t}}{1+W_t^2}\right)dW_t \\ &+ \left(\arctan(W_t)\frac{e^{W_t}}{2} - \frac{W_te^{W_t}}{(1+W_t^2)^2} + \frac{e^{W_t}}{1+W_t^2}\right)dt \\ &(= d(X_tY_t)) \end{aligned}$$

0026-2: a) $W_\bullet$ is Brownian motion, and $X_t = \arctan(t)$ and so

$$dX_t = \frac{1}{1+t^2}dt$$

b) $Yt = e^{W_t}$ and so

$$dY_t = e^{W_t}dW_t + \frac{e^{W_t}}{2}dt$$

c) By Ito's Lemma (second version), we get

$$\begin{aligned}
d(X_tY_t) &= d(\arctan(t)e^{W_t})(= f(W_t,t)) \\
&= \left[e^{W_t}\arctan(t)\right]dW_t + \left[\frac{e^{W_t}}{2}\arctan(t)\right]dt + \left[\frac{e^{W_t}}{1+t^2}\right]dt \\
&= \left[e^{W_t}\arctan(t)\right]dW_t + \left[\frac{e^{W_t}}{2}\arctan(t) + \frac{e^{W_t}}{1+t^2}\right]dt
\end{aligned}$$

d) From **a), b) and c)**, we get

$$[(X_t)(dY_t)] = (\arctan(t))\left(e^{W_t}dW_t + \frac{e^{W_t}}{2}dt\right)$$

$$[(dX_t)(Y_t)] = \left(\frac{1}{1+t^2}\right)e^{W_t}$$

and so

$$\begin{aligned}
[(X_t)(dY_t)] + [(dX_t)(Y_t)] &= [\arctan(t)e^{W_t}]dW_t + \left[\frac{e^{W_t}}{2}\arctan(t) + \frac{e^{W_t}}{1+t^2}\right]dt \\
&(= d(X_tY_t))
\end{aligned}$$

0026-3: We suppose that $dX_t = \sigma_t dW_t + \mu_t dt$. Then define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = e^x$ (i.e. f is constant on y for fixed x), and let $U_t := f(X_t,t)$. Then

$$\begin{aligned}
\frac{d(e^{X_t})}{e^{X_t}} &= \frac{1}{e^{X_t}}dU_t \\
\text{(Ito's Lemma 3 gives)} \quad &= \frac{1}{e^{X_t}}\left[e^{X_t}dX_t + \frac{e^{X_t}}{2}(dX_t)^2 + 0dt\right] \\
&= \sigma_t dW_t + \mu_t dt + (\sigma_t dW_t + \mu_t dt)^2/2 \\
&= \sigma_t dW_t + \mu_t dt + \sigma_t^2/2dt
\end{aligned}$$

0026-4: a) $X_t = e^{4W_t+t^3}$ and if we let $f(x,t) = e^{4x+t^3}$, then $X_t = f(W_t,t)$. Then using Ito's Lemma, we get

$$\begin{aligned}
dX_t &= 4e^{4W_t+t^3}dW_t + 8e^{4W_t+t^3}dt + 3t^2e^{4W_t+t^3}dt \\
&= 4e^{4W_t+t^3}dW_t + \left[8e^{4W_t+t^3} + 3t^2e^{4W_t+t^3}\right]dt
\end{aligned}$$

b) From part **a)**, we have that

$$dX_t = 4e^{4W_t+t^3}dW_t + \left[8e^{4W_t+t^3} + 3t^2e^{4W_t+t^3}\right]dt$$

Define

$$\sigma(X_t,t) := 4X_t$$

and

$$\mu(X_t, t) := 8X_t + 3t^2 X_t$$

Then

$$dX_t = \sigma(X_t, t)dW_t + \mu(X_t, t)dt$$

Now if $Y_t = \sin(X_t + t)$ and we define $g(x, t) := \cos(x + t)$, then $Y_t = g(X_t, t)$. Thus by applying the third version of Ito's Lemma, we get

$$\begin{aligned} dY_t = {} & [(\partial_1 g)(X_t, t)][\sigma(X_t, t)]dW_t \\ & + [(\partial_1 g)(X_t, t)][\mu(X_t, t)]dt \\ & + [(\partial_1^2 g)(X_t, t)/2][\sigma(X_t, t)^2]dt \\ & + [(\partial_2 g)(X_t, t)]dt \end{aligned}$$

or

$$\begin{aligned} dY_t = {} & [\cos(X_t + t)4X_t]\, dW_t \\ & + \left[\cos(X_t + t)(8X_t + 3t^2 X_t) - \frac{\sin(X_t + t)}{2}(4X_t)^2 + \cos(X_t + t)\right] dt \end{aligned}$$

If we want, we could replace X_t by $e^{4W_t + t^3}$ to remove all traces of X_t, however this answer should suffice.

0026-5: We suppose that $X_\bullet$ solves

$$dX_t = X_t(0.25dW_t + 0.06dt), \qquad X_0 = 4$$

Set$Y_t := \ln X_t$. Then

$$\begin{aligned} dY_t &= \frac{dX_t}{X_t} - \frac{1}{2}\left(\frac{dX_t}{X_t}\right)^2 \\ &= 0.25dW_t + 0.06dt - 0.5(0.25dW_t + 0.06dt)^2 \\ &= 0.25dW_t + 0.06dt - 0.5(.0625)dt \\ &= .025dW_t + 0.02875dt \end{aligned}$$

Hence

$$Y_t = 0.25W_t + 0.02875t + C$$

Using our initial condition that $X_0 = 4$ we get $Y_0 = \ln 4$, and so $C = \ln 4$. Since $Y_t = \ln X_t$, we get that

$$\begin{aligned} X_t &= e^{Y_t} \\ &= e^{0.25W_t + 0.02875t + \ln 4} \\ &= 4e^{0.25W_t + 0.02875t} \end{aligned}$$

0026-6: a) $dX_t = 0.4dW_t - 0.05X_t dt$, and so if $\sigma(V_t, t) := 0.4$ and $\mu(X_t, t) := -0.05X_t$, then X_t is a solution of $dX_t = \sigma dW_t + \mu dt$. Now $U_t = e^{0.05t}X_t$, and so if we define $f(x, t) := e^{0.05t}x$, then $U_t = f(X_t, t)$. Therefore by version three of Ito's Lemma, we have

$$dU_t = 0.4e^{0.05t}dW_t - 0.05X_t e^{0.5t}dt + 0dt + 0.05X_t e^{0.05t}dt$$

or

$$\boxed{dU_t = 0.4e^{0.05t}dW_t; \qquad U_0 = 3}$$

Now $V_t = U_t^2$, so if we set $g(x,t) := e^{0.1t}X_t^2$, then $V_t = g(X_t, t)$. Thus,

$$dV_t = 2e^{0.1t}X_t \cdot 0.4dW_t + 2e^{0.1t}X_t(-0.05X_t)dt + e^{0.1t}(0.4)^2dt + 0.1e^{0.1t}X_t^2dt$$

or

$$\boxed{dV_t = 0.8e^{0.1t}X_tdW_t + 0.16e^{0.1t}dt}$$

b) Since

$$dU_t = 0.4e^{0.05t}dW_t$$

we get

$$U_9 - U_0 = 0.4\int_0^9 e^{0.05t}dW_t$$

or

$$U_9 = 3 + 0.4\int_0^9 e^{0.05t}dW_t$$

Since $U_t = e^{0.05t}X_t$, we get that

$$X_9 = e^{-0.45}U_9$$

and so

$$X_9 = e^{-0.45}\left[3 + 0.4\int_0^9 e^{0.05t}dW_t\right]$$

Now

$$0.4\int_0^9 e^{0.05t}dW_t \sim \mathcal{N}(0, (0.4)^2\int_0^9 e^{0.1t}dt)$$

or

$$0.4\int_0^9 e^{0.05t}dW_t \sim \mathcal{N}(0, 1.6(e^{0.9} - 1))$$

Therefore,

$$X_9 \sim \mathcal{N}(3e^{-0.45}, 1.6(e^{0.9} - 1))$$

Finally,

$$\boxed{E[U_9] = E[e^{0.45}X_9] = 3e^{0.45}e^{-0.45} = 3}$$

Similarly, since

$$dV_t = 0.8e^{0.1t}X_tdW_t + 0.16e^{0.1t}dt$$

we have that

$$V_9 - V_0 = \int_0^9 0.8e^{0.1t}X_tdW_t + 0.16e^{0.1t}dt$$

or

$$V_9 = V_0 + \int_0^9 0.16e^{0.1t}dt$$

This then gives, with $V_0 = 9$

$$\boxed{V_9 = 9 + 1.6(e^{0.9} - 1)}$$

d)

$$Var[X_9] = E[e^{-0.9}V_9] - e^{-0.9}E[U_9]^2$$

or

$$Var[X_9] = 1.6(1 - e^{-0.9})$$

Topic 27

0027-1: a) $X_t = e^{4W_t+t^3}$ so

$$dX_t = 4e^{4W_t+t^3}dW_t + (8+3t^2)e^{4W_t+t^3}dt$$

Thus define σ_t and γ_t by

$$\sigma_t := 4e^{4W_t+t^3} \quad \text{and} \quad \gamma_t := 2 + (3/4)t^2$$

b) First we compute:

$$\frac{1}{2}\int_0^T \gamma_t^2 dt = 2T + T^3/2 + 9/160T^5$$

Hence,

$$E\left[e^{\frac{1}{2}\int_0^T \gamma_t^2 dt}\right] = e^{2T+T^3/2+(9/160)T^5}$$

c) This follows precisely from the fact that the value in **b)** is finite and then applying Girsanov's Theorem.

d)

$$\begin{aligned}
E\left[\int_0^T \sigma_t^2 dt\right] &= \int_0^T E[\sigma_t^2]dt \\
&= \int_0^T E\left[16e^{8W_t+2t^3}\right]dt \\
\text{(since } W_t \sim \sqrt{t}Z) \qquad &= \int_0^T \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 16e^{8\sqrt{t}x++2t^3}e^{-x^2/2}dx\right]dt \\
&= \int_0^T \left[\frac{16e^{2t^3}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{32t}e^{-(x-8\sqrt{t})^2/2}dx\right]dt \\
&= \int_0^T 16e^{2t^3+32t}dt \\
&< \infty
\end{aligned}$$

e) This follows from the fact on slide 9 of topic 27.

0027-2: a) The two terms inside the exponential in the expansion of ζ_T are

$$\int_0^T -\gamma_s dW_s - \frac{\gamma_s^2}{2}ds$$

b) The two terms inside the exponential in the expansion of $\exp(-i\int_0^T \xi_s d\tilde{W})$ are

$$-i\int_0^T \xi_s dW_s + \xi_s\gamma_s ds$$

c) The four terms inside the exponential in the expansion of $\zeta_T\cdot \exp(-i\int_0^T \xi_s d\tilde{W}_s)$ is

$$\int_0^T -\gamma_s dW_s - \frac{\gamma_s^2}{2}ds - i\int_0^T \xi_s dW_s + \xi_s\gamma_s ds$$

or

$$\int_0^T -\gamma_s dW_s - \frac{\gamma_s^2}{2}ds - i\xi_s dW_s - i\xi_s\gamma_s ds$$

d) The five terms inside the exponential in the expansion of ρ_T are

$$\int_0^T -\gamma_s dW_s - \frac{\gamma_s^2}{2}ds - i\xi_s dW_s - i\xi_s\gamma_s ds + \frac{\xi_s^2}{2}ds$$

Topic 28

0028-1: a) There is an error in the statement of the problem. It should read "Find $\sigma_\bullet$ and $\gamma_\bullet$ s.t. $dX_t = \sigma_t(dW_t + \gamma_t \underline{dt})$" Hence,

$$\sigma_t = 0.3X_t \qquad \text{and} \qquad \gamma_t = 0.2$$

b) If we set $Y_t := \ln(X_t)$ we get

$$\begin{aligned}
dY_t &= \frac{dX_t}{X_t} - \frac{1}{2}\left(\frac{dX_t}{X_t}\right)^2 \\
&= (0.3dW_t + 0.06dt) - 0.5(0.09)dt \\
&= 0.3dW_t - 0.015dt \\
\text{(and so)} \qquad Y_t &= 0.3W_t - 0.015t + \ln 7 \\
\text{(and since } Y_t = \ln(X_t)) \qquad X_t &= e^{0.3W_t - 0.015t + \ln 7} \\
\text{(or)} \qquad X_t &= 7e^{0.3W_t - 0.015t}
\end{aligned}$$

c) $\gamma_t = 0.2$ and so

$$\begin{aligned}
E\left[e^{\frac{1}{2}\int_0^T \gamma_t^2 dt}\right] &= e^{0.5\int_0^T E[\gamma_t^2]dt} \\
&= e^{0.5\int_0^T E[0.2^2]dt} \\
&= e^{0.5\int_0^T 0.04dt} \\
&= e^{0.002T} \\
&< \infty
\end{aligned}$$

0028-2: a) Here we have

$$dX_t = 0.2dW_t + 0.035dt, \qquad X_0 = 4$$
$$dB_t = 0.05B_t dt$$
$$S_t = e^{X_t}$$

and we get that

$$\begin{aligned}\frac{dS_t}{S_t} &= dX_t + \frac{1}{2}dX_t^2\\ &= 0.2dW_t + 0.055dt\\ &= 0.2d\tilde{W}_t + 0.05dt\end{aligned}$$

where $d\tilde{W}_t = dW_t + \frac{0.005}{0.2}dt$. Solving the SDE using the initial condition that $S_0 = e^4$, we get

$$S_t = e^4 e^{0.2\tilde{W}_t + 0.03t}$$

Hence, the price of the derivative that pays $(6 - e^{X_7})_+$ at time $t = 7$ is given by

$$\begin{aligned}P &= e^{-0.05\cdot 7}E^{\nu}[(6 - e^{X_7})_+]\\ &= e^{-0.05\cdot 7}E^{\nu}[(6 - S_7)_+]\\ &= e^{-0.35}/\sqrt{2\pi}\int_{-\infty}^{\infty}(6 - e^4 e^{0.2\sqrt{7}x+0.21})_+ e^{-x^2/2}dx\\ &= e^{-0.35}/\sqrt{2\pi}\int_{-\infty}^{-4.213}(6 - e^4 e^{0.2\sqrt{7}x+0.21})e^{-x^2/2}dx\\ &= 5.489\times 10^{-6}\end{aligned}$$

b) We want to find the price as a function of x and t, and then take the partial derivative with respect to x to find $F(x,t)$. So take S_t to be x and recalculate. Thus we get

$$S_t = xe^{0.2\tilde{W}_t + 0.03t}$$

and hence,

$$\begin{aligned}P(x,t) &= e^{-0.05(7-t)}E^{\nu}[(6 - S_7)_+]\\ &= \frac{e^{-0.05(7-t)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(6 - xe^{0.2\sqrt{7}s+0.21})_+ e^{-s^2/2}ds\\ &= \frac{e^{-0.05(7-t)}}{\sqrt{2\pi}}\int_{-\infty}^{2.989-\ln x/(.2\sqrt{7})}(6 - xe^{0.2\sqrt{7}s+0.21})e^{-s^2/2}ds\end{aligned}$$

(and so) $$F(x,t) = \left(\frac{\partial}{\partial x}\right)\left[\frac{e^{-0.05t}}{\sqrt{2\pi}}\int_{-\infty}^{2.989-\ln x/(.2\sqrt{7})}(6 - xe^{0.2\sqrt{7}s+0.21})e^{-s^2/2}ds\right]$$