

Topic 11

0011-1: Note that

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

Thus,

$$\lambda(\mathbb{R}) = \lambda(\mathbb{Q}) + \lambda(\mathbb{R} \setminus \mathbb{Q})$$

Since $\lambda(\mathbb{Q}) = 0$ (because \mathbb{Q} is a countable set) and $\lambda(\mathbb{R}) = \infty$, we have

$$\lambda(\mathbb{R} \setminus \mathbb{Q}) = \infty$$

0011-2: The Lebesgue measure of $[0, 1/3] \setminus (1/9, 2/9)$ is equal to the Lebesgue measure of $[0, 1/9] \cup [2/9, 1/3]$, which is $2/9$.

0011-3: The Lebesgue measure of this set is the Lebesgue measure of each minus the Lebesgue measure of the intersection. This turns out to be equal to 16 .

0011-4: The first thing to note is that these sets are piecewise disjoint, so the measure of the union is equal to the sum of the measures, i.e.

$$\begin{aligned} \lambda\left(\bigcup_{n=1}^{\infty} [n, n + 2^{-n}] \times [n, n + 3^{-n}]\right) &= \sum_{n=1}^{\infty} \lambda\left([n, n + 2^{-n}] \times [n, n + 3^{-n}]\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^n \\ &= \frac{1}{6} \sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n \\ &= \frac{1}{6} \frac{1}{1 - \frac{1}{6}} \\ &= \frac{1}{5} \end{aligned}$$

0011-5: Note that X is a probability space, and since $Y = X \times X \times X \times \dots$, Y is a probability space. Now, S is defined to be the set of all $y \in Y$ such that $p_2(y) = p_4(y) = p_6(y) = \dots$. To compute $\nu(S)$, it is useful to think in terms of probability, i.e. $\nu(S)$ is the probability that every other coin flip comes up the same in a infinite string of coin flips. Let's start with an easier example first, before calculating $\nu(S)$. Let T be defined to be the set of all $y \in Y$ such that the first two coin flips are heads, so

$$T = \{(H, H, x_3, x_4, \dots) | x_i \in X\}$$

Then $\nu(T)$ is the probability that in an infinite string of coin flips, the first two flips come up heads. Thus,

$$\begin{aligned}\nu(T) &= \mu(\{H\}) \cdot \mu(\{H\}) \cdot \mu(\{H, T\}) \cdot \mu(\{H, T\}) \dots \\ &= Pr[x_1 = H] \cdot Pr[x_2 = H] \cdot Pr[x_3 = H \text{ or } T] \cdot Pr[x_4 = H \text{ or } T] \dots \\ &= (.5)(.5)(1)(1) \dots \\ &= .25\end{aligned}$$

If T' is now the set of all $y \in Y$ such that the first two coin flips are the same, then $\nu(T') = .5$, for there is a 25% probability that the first two flips are heads and a 25% probability that the first two flips are tails.

Now, back to our original problem. Using the same reasoning, we calculate

$$\begin{aligned}\nu(S) &= \mu(\{H, T\}) \cdot \mu(\{H\}) \cdot \mu(\{H, T\}) \cdot \mu(\{H\}) \dots \\ &\quad + \mu(\{H, T\}) \cdot \mu(\{T\}) \cdot \mu(\{H, T\}) \cdot \mu(\{T\}) \dots \\ &= Pr[x_1 = H \text{ or } T] \cdot Pr[x_2 = H] \cdot Pr[x_3 = H \text{ or } T] \cdot Pr[x_4 = H] \cdot \dots \\ &\quad + Pr[x_1 = H \text{ or } T] \cdot Pr[x_2 = T] \cdot Pr[x_3 = H \text{ or } T] \cdot Pr[x_4 = T] \cdot \dots \\ &= (1)(.5)(1)(.5) \dots + (1)(.5)(1)(.5) \dots \\ &= \prod_{i=1}^{\infty} (.5)^i + \prod_{j=1}^{\infty} (.5)^j \\ &= 0 + 0 = 0\end{aligned}$$

This should make sense intuitively. If you start a sequence of coin flips, the probability that the second flip will match the fourth flip, which will match the sixth flip, which will match the eighth flip, etc. decreases to 0 as you keep flipping coins.

0011-6: The minimal sets are

$$[0, 20); \quad [20, 50); \quad [50, 115); \quad [115, 300)$$

0011-7: a) The key here is that σ -algebras contain countable unions and countable intersections of its elements. Now, π has an infinite decimal expansion

$$\pi = 3.14159265359 \dots$$

So define

$$\begin{aligned}A_1 &= (2, 4) \\ A_2 &= (3.1, 3.2) \\ A_3 &= (3.14, 3.15) \\ A_4 &= (3.141, 3.142) \\ A_5 &= (3.1415, 3.1416)\end{aligned}$$

⋮

Notice that each A_n is in the σ -algebra, being an interval of rational numbers, and these are nested intervals, i.e. $A_n \subset A_{n+1}$ which shrink down to π . Thus

$$\bigcap_{n=1}^{\infty} A_n = \{\pi\}$$

The same idea applies to $\sqrt{\pi} = 1.77245385091\dots$. Thus, since $\{\pi\}$ and $\{\sqrt{\pi}\}$ are in S , so is their union, $\{\pi, \sqrt{\pi}\} \in S$.

b) In a similar fashion, let

$$\begin{aligned} A_1 &= (2, 3) \\ A_2 &= (1.8, 3.1) \\ A_3 &= (1.78, 3.14) \\ A_4 &= (1.773, 3.141) \\ A_5 &= (1.7725, 3.1415) \\ &\vdots \end{aligned}$$

Then

$$\bigcup_{n=1}^{\infty} A_n = (\sqrt{\pi}, \pi) \in S$$

c) Now let

$$\begin{aligned} A_1 &= (1, 4) \\ A_2 &= (1.7, 3.2) \\ A_3 &= (1.77, 3.15) \\ A_4 &= (1.772, 3.142) \\ A_5 &= (1.7724, 3.1416) \\ &\vdots \end{aligned}$$

Then

$$\bigcap_{n=1}^{\infty} A_n = [\sqrt{\pi}, \pi] \in S$$