Topic 21

0021-1: Here, $\sigma = 9$, $n = 75$, and $\overline{X} = 82$. Thus,

$$\begin{aligned} .95 &= Pr\left[\left|\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right| \leq 1.96\right] \\ &= Pr\left[\left|\frac{82-\mu}{9/\sqrt{75}}\right| \leq 1.96\right] \\ &= Pr\left[|\mu - 82| \leq 2.037\right] \end{aligned}$$

Thus our 95% confidence interval is $(82 - 2.037, 82 + 2.037)$ or $(79.963, 84.037)$.

Topic 22

0022-1: a) For any rectangle $I \times J \subset \mathbb{R}^2$, let $g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ be a PDF on I and $g(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$ be a PDF on J. Then the joint PDF is

$$f(x,y) := g(x) \times g(y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

Then,

$$\begin{aligned} f\nu(I \times J) &= \int_{I\times J} f d(\lambda \times \lambda) \\ &= \int_I \int_J f(x,y) d\lambda(y) d\lambda(x) \\ &= \int_I g(x) d\lambda(x) \int_J g(y) d\lambda(y) \\ &= (g\lambda)(I)(g\lambda)(J) \\ &= (Z_*(I))(Z_*(J)) \\ &= (\lambda_1 \times \lambda_1)(Z^{-1}(I) \times Z^{-1}(J)) \\ &= (\lambda_1 \times \lambda_1)(X^{-1}(I \times J) \\ &= X_*(\lambda_1 \times \lambda_1)(I \times J) \\ &= \mu(I \times J) \end{aligned}$$

Hence, $\mu = f\nu$, and we have

$$\frac{d\mu}{d\nu} = \frac{1}{2\pi}e^{-(x^2+y^2)/2}$$

b) The sum of two normal variables is another normal variable with its mean and variance equal to the sum of means and the sum of variance respectively. So the PDF of Y is

$$h(x) = \frac{1}{2\sqrt{\pi}}e^{-x^2/4}$$

Then for $I = [a, b] \subset \mathbb{R}$ we have

$$\begin{aligned}
\tau(I) &= (\lambda_1 \times \lambda_1)(Y^{-1}(I)) \\
&= (\lambda_1 \times \lambda_1)\{(s,t)|Z(s) + Z(t) \in I\} \\
&= (\lambda_1 \times \lambda_1)X^{-1}\{(p,q)|p + q \in I\} \\
&= \mu(\{(p,q)|p + q \in I\}) \\
&= f\nu(\{(p,q)|p + q \in I\}) \\
&= \int_{\mathbb{R}} \int_{[a-p,b-p]} f(p,q)dqdp \\
&= \int_{\mathbb{R}} \int_{[a,b]} f(p,p+q)dqdp \\
&= \int_{\mathbb{R}} \int_{I} \frac{1}{2\pi} e^{-(p^2+(p+q)^2)/2} dqdp \\
&= \int_{I} \int_{\mathbb{R}} \frac{1}{2\pi} e^{-(p^2+(p+q)^2)/2} dpdq \\
&= \frac{1}{2\pi} \int_{I} \int_{\Re}^{-(2p^2+2pq+q^2)/2} dpdq \\
&= \frac{1}{2\pi} \int_{I} e^{-q^2/4} dq \int_{\mathbb{R}} e^{-((\sqrt{2}(p+q/2))^2)/2} dp \\
&= \frac{1}{2\pi} \int_{I} e^{-q^2/4} dq \int_{\mathbb{R}} \frac{1}{\sqrt{2}} e^{-p^2/2} dp \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{2}} \int_{I} e^{-q^2/4} dq \int_{\mathbb{R}} e^{-p^2/2} dp \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{2}} 2\sqrt{\pi} h\lambda(I)\sqrt{2\pi} \\
&= h\lambda(I)
\end{aligned}$$

Hence, $\frac{d\tau}{d\lambda} = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$.

0022-2: Here $n = 75$ so there are 74 degrees of freedom. The sample standard deviation is $s = 8$ so $s^2 = 64$. Obtaining the values for the .005 and .995 percentiles of the χ^2 distribution table for 74 degrees of freedom, we calculate:

$$Pr[\frac{74s^2}{\sigma^2} > 109.074] = .005$$

and

$$Pr[\frac{74s^2}{\sigma^2} < 46.417] = .005$$

Hence,

$$Pr[\frac{74 \cdot 64}{109.074} > \sigma^2] = .005 \quad \text{and} \quad Pr[\frac{74 \cdot 64}{46.417} < \sigma^2] = .005$$

or

$$Pr[43.42 > \sigma^2] = .005 \quad \text{and} \quad Pr[102.032 < \sigma^2] = .005$$

and so

$$Pr[6.589 > \sigma] = .005 \quad \text{and} \quad Pr[10.101 < \sigma^2] = .005$$

Thus, the 99% confidence interval for σ is $(6.589, 10.101)$.

Topic 23

0023-1: a)

$$Pr[X < x] = \begin{cases} 1 - e^{-\alpha x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Now to solve for α, we write

$$\begin{aligned} 10 = E[X] &= \int_{\mathbb{R}} x\alpha e^{-\alpha x} dx \\ &= \int_0^\infty x\alpha e^{-\alpha x} dx \\ &= -xe^{-\alpha x}|_0^\infty + \int_0^\infty e^{-\alpha x} dx \\ &= -\frac{1}{\alpha} e^{-\alpha x}|_0^\infty \\ &= \frac{1}{\alpha} \end{aligned}$$

Thus, $\alpha = 0.1$. Now we solve

$$\begin{aligned} Pr[X \geq 12] &= 1 - Pr[X < 12] \\ &= 1 - (1 - e^{-\alpha \cdot 12}) \\ &= e^{-.1 \cdot 12} \\ &= e^{-1.2} \end{aligned}$$

b) Since U, V, and W are iids with the same distribution as X, the PDF for $U + V + W$ is

$$(*^3 \epsilon_\alpha)(x) = \frac{0.1^3 x^2 e^{-.1x}}{2!} = .0005 x^2 e^{-.1x}$$

Thus, the CDF is given by

$$\begin{aligned}
CDF_{\delta_{U+V+W}}(x) &= \int_0^x .0005t^2e^{-.1t}dt \\
&= .0005\int_0^x t^2e^{-.1t}dt \\
&= .0005\left[-t^2 10e^{-.1t}|_0^x + \int_0^x 20te^{-.1t}dt\right] \\
&= .0005\left[-10x^2e^{-.1x} + 20\left(-10te^{-.1t}|_0^x + 10\int_0^x e^{-.1t}dt\right)\right] \\
&= .0005\left[-10x^2e^{-.1x} - 200xe^{-.1x} - 2000e^{-.1x} + 2000\right] \\
\text{(and hence)}\quad CDF_{\delta_{U+V+W}}(35) &= .0005\left[-10\cdot 35^2e^{-3.5} - 200\cdot 35e^{-3.5} - 2000e^{-3.5} + 2000\right] \\
&= .6792 \\
&= Pr[U+V+W < 35] \\
\text{(and so)}\quad Pr[U+V+W \geq 35] &= 1 - .6792 = \boxed{.3208}
\end{aligned}$$

c) We have the same value for α again, as Y_i has the same distribution as X. Now

$$Pr[N_{100} = k] = e^{-\alpha t}\frac{(\alpha t)^n}{n!} = e^{-10}\frac{(10)^k}{k!}$$

and so

$$\begin{aligned}
E[N_{100}] &= \sum_{k=0}^{\infty} ke^{-10}\frac{(10)^k}{k!} \\
&= e^{-10}\sum_{k=1}^{\infty} 10\frac{10^{k-1}}{(k-1)!} \\
&= e^{-10}10\sum_{k=0}^{\infty}\frac{10^k}{k!} \\
&= e^{-10}10e^{10} \\
&= 10
\end{aligned}$$

0023-3: a) Here, $\mu = \dfrac{4\delta_0 + 5\delta_1 + \delta_2}{14}$
Thus, $\mu(\{0\}) = 4/14$

b) $\mu(\{1\}) = 5/14$

c) $\mu(\{2\}) = 5/14$

d) $\mu(\{3\}) = 0$

e)

$$\int_{-\infty}^{\infty} x d\mu(x) = 0 \cdot \frac{4}{14} + 1 \cdot \frac{5}{14} + 2 \cdot \frac{5}{14} = \frac{15}{14}$$

f)

$$\int_{-\infty}^{\infty} x^2 d\mu(x) = 0^2 \cdot \frac{4}{14} + 1^2 \cdot \frac{5}{14} + 2^2 \cdot \frac{5}{14} = \frac{25}{14}$$