Final Exam Solutions

I. Definitions:

a) Let $(\mathcal{M}, \mathcal{B})$ be a Borel space. Then a measure on $(\mathcal{M}, \mathcal{B})$ is a function $\mu : \mathcal{B} \to [0, \infty]$ such that ...

$$\mu \text{ is } \sigma\text{-additive}$$

b) Let μ be a measure of $\mathbb{R}$. The cumulative distribution function of μ is the function $f : \mathbb{R} \to [0, 1]$ defined by $f(x) = \ldots$

$$\mu((-\infty, x])$$

c) Let $\mathcal{B}$ be a σ-algebra on a set M and let $\mathcal{C}$ be a σ-algebra on a set N. Let $\mathcal{A} := B \times C | B \in \mathcal{B}, C \in \mathcal{C}$. Then $B \times C := \ldots$

$$< \mathcal{A} >_\sigma$$

d) Let μ be a measure on $(M, \mathcal{B})$ and let ν be a measure on $(M, \mathcal{B})$. Then μ is equivalent to ν if ...

$$\mu << \nu \quad \text{and} \quad \nu << \mu$$

e) Let λ be Lebesgue measure on $\mathbb{R}$. Let μ be a probability measure on $\mathbb{R}$. A probability density function for μ is a function $f : R \to \mathbb{R}$ such that ...

$$\mu = f\lambda$$

f) Let X and Y be L^2 random variables. Then $Corr[X, Y] = \ldots$

$$\frac{Cov[X, Y]}{\sqrt{Var[X]}\sqrt{Var[Y]}}$$

g) Let $\mathcal{F}_\bullet$ be a filtration and let $X_\bullet$ be a process. Then $X_\bullet$ is $\mathcal{F}_\bullet$-adapted if ...

$$\forall t \in [0, \infty), X_t \text{ is } \mathcal{F}_t\text{-measureable}$$

h) Let $\mathcal{B}^1_\bullet, \mathcal{B}^2_\bullet, \mathcal{B}^3_\bullet, \ldots$ be the standard Brownian walks, as defined in class. A process $\mathcal{W}_\bullet$ is a Brownian motion if ...

$$\mathcal{B}^1_\bullet, \mathcal{B}^2_\bullet, \mathcal{B}^3_\bullet, \ldots \to \mathcal{W}_\bullet \text{ in f.d. marginals and } \mathcal{W}_\bullet \text{ is continuous}$$

II. True or False

a) Let $W_\bullet$ be a Brownian motion and let Z be a standard normal random variable. Then, for all $t \geq 0$, the distribution of W_t is equal to the distribution of tZ.

False

b) If two random variables have the same distribution, then they cannot be independent.

False

c) Any Brownian motion is a martingale.

True

d) Let $W_\bullet$ be a Brownian motion, defined on a probability space $(\Omega, \mathcal{B}, \mu)$. Then, for almost every $\omega \in \Omega$, the Brownian path $W_\bullet(\omega)$ is of bounded variation.

False

e) The composition of two Borel functions is Borel.

True

f) Let X be a random variable. If the Fourier transform of $\delta[X]$ is $e^{-t^2/2}$, then X is standard normal.

True

g) Let X and Y be independent random variables. Let p be a probability density function of $\delta[X]$ and let q be a probability density function of $\delta[Y]$. Then $p+q$ is a probability density function of $\delta[X+Y]$.

False

h) For all $x \in \mathbb{R}$, let δ_x denote a point mass at x. Then $\delta_{1/n} \to \delta_0$, as $n \to \infty$.

True

III. Computations

1. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$. Let λ denote Lebesgue measure on $\mathbb{R}$ and let $\mu := h\lambda$. Find a probability density function for $\mu * \mu$. SOLUTION:

$$\begin{aligned}
PDF_{\mu*\mu}(x) &= \int_{-\infty}^{\infty} h(t)h(x-t)dt \\
&= \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}} \frac{e^{-(x-t)^2/2}}{\sqrt{2\pi}} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(t^2+x^2+t^2-2xt)^2/2} dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(\sqrt{t}-\sqrt{x/2})^2/2-x^2/4} dt \\
\text{(let } u = \sqrt{t} \text{ to get)} \qquad &= \frac{e^{-x^2/4}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{2}} e^{-(u-\sqrt{x/2})^2/2} du \\
&= \frac{e^{-x^2/4}}{\sqrt{2\pi}\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e-(u-\sqrt{x/2})^2/2du \\
&= \frac{e^{-x^2/4}}{\sqrt{2\pi}\sqrt{2}}
\end{aligned}$$

2. Let $g(x) = x^5 + 4$. Let $v(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$. Compute $\int_{-1}^{1} g(x)dv(x)$.
SOLUTION: Note that

$$dv = 2xd\lambda + 2\delta_0$$

And so,

$$\begin{aligned}
\int_{-} 1^1 x^5 + 4dv &= \int_{-1}^{1} (x^5+4)2xd\lambda + \int_{-1}^{1} (x^5+4)2d\delta_0 \\
&= \frac{4}{7} + 8 \\
&= \frac{60}{7}
\end{aligned}$$

3. Let X have a Student t distribution with two degrees of freedom, so that a probability density function for $\delta[X]$ is

$$p(x) = \frac{1}{2\sqrt{2}} \left(1 + \frac{x^2}{2}\right)^{-3/2}$$

Compute a probability density function of $\delta[X^2]$.
SOLUTION:

$$\begin{aligned} Pr[X^2 < t] &= Pr[-\sqrt{x} < X < \sqrt{x}] \\ &= CDF_X(\sqrt{x}) - CDF_X(\sqrt{x}) \end{aligned}$$

(and so)
$$\begin{aligned} PDF &= PDF_X(\sqrt{x})\frac{1}{2\sqrt{x}} - PDF_X(-\sqrt{x})\frac{-1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}}\left[\frac{1}{2\sqrt{2}}\left(1+\frac{(\sqrt{x})^2}{2}\right)^{-3/2} + \frac{1}{2\sqrt{2}}\left(1+\frac{(-\sqrt{x})^2}{2}\right)^{-3/2}\right] \\ &= \frac{1}{2\sqrt{2}}\left(1+\frac{x}{2}\right)^{-3/2}\frac{1}{\sqrt{x}} \end{aligned}$$

4. Compute $E\left[\int_0^3 W_t^8 dt\right]$.
SOLUTION:

$$\begin{aligned} E\left[\int_0^3 W_t^8 dt\right] &= \int_0^3 E\left[W_t^8\right] dt \\ &= \int_0^3 \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(\sqrt{t}x)^8 e^{-x^2/2}\right] dt \\ &= \int_0^3 \left[t^4 \cdot 7 \cdot 5 \cdot 3 \cdot 1\right] dt \\ &= 7 \cdot 3^6 \end{aligned}$$

5. Compute $E\left[\int_0^3 W_t^4 dW_t\right]$.
SOLUTION: 0

6. Let $W_\bullet$ be a Brownian motion. Let $X := \int_0^3 t^3 dW_t$. Compute $E[X^4]$.
SOLUTION:

$$\int_0^3 t^6 dt = \frac{3^7}{7}$$

Thus, $X \sim N(0, \frac{3^7}{7})$. And so,

$$\begin{aligned} E[X^4] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{3^7}{7}}x\right)^4 e^{-x^2/2}dx \\ &= (\frac{3^7}{7})^2 \cdot 3 \cdot 1 \\ &= \frac{3^{15}}{49} \end{aligned}$$

7. Let $\Omega := [0,1] \times [0,1]$, with its standard σ-algebra and with Lebesgue measure. Let $U : \Omega \to \mathbb{R}$ be the random variable defined by $U(x,y) = x$. Let $V : \Omega \to \mathbb{R}$ be the random variable defined by $V(x,y) = 2e^x y$. Compute $E[V|U]$.
SOLUTION:

$$\begin{aligned} E[V|U](x,y) &= \int_0^1 2e^x y dy \\ &= 2e^x \frac{1}{2} \\ &= e^x \end{aligned}$$

8. Let $X_1, \ldots, X_{100}$ be iid normal variables with unknown mean μ and known variance 0.5. Let $x_1, \ldots, x_{100}$ be a sample modeled on $X_1, \ldots, X_{100}$. Assume that the sample mean $(x_1 + \cdots + x_{100})/100 = 2$. Find a 95% confidence interval for μ.
SOLUTION:

$$Var[X_1, \ldots, X_{100}] = 0.5 \cdot 100 = 50$$

and so

$$Var[(X_1, \ldots, X_{100})/100] = 50/100^2 = 1/200 = 0.005$$

Hence,

$$\left| \frac{2-\mu}{\sqrt{0.005}} \right| < 1.96$$

and so

$$|2-\mu| < 1.96\sqrt{0.005}$$

and so

$$2 - 1.96\sqrt{0.005} < \mu < 2 + 1.96\sqrt{0.005}$$

9. Let $W_\bullet$ be a Brownian motion. Let $X_\bullet$ satisfy $dX_t = 4t^3 dW_t + 2tdt$ with initial condition $X_0 = 3$. Compute $Var[X_7]$.
SOLUTION:

$$\begin{aligned} X_t - X_0 &= \int_0^t 4_s^3 dW_s + \int_0^t 2sds \\ X_t &= 3 + \int_0^t 4s^3 dW_s + t^2 \\ &= 3 + t^2 + \int_0^t 4s^3 dW_s \end{aligned}$$

and so $X_t \sim N(3+t^2, \int_0^t (4s^3)^2 ds)$. Thus,

$$\begin{aligned} Var[X_7] &= \int_0^7 (4s^3)^2 ds \\ &= 16 \cdot 7^6 \end{aligned}$$

10. Let $W_\bullet$ be a Brownian motion. Let $X_\bullet$ satisfy the Ornstein-Uhlenbeck SDE $dX_t = 2dW_t - (1/2)X_t dt$ with initial condition $X_0 = 3$. Compute $\mathrm{Var}[X_7]$.
SOLUTION:

$$Y_t = e^{0.5t} X_t \qquad Y_0 = 3$$

Now

$$\begin{aligned} dY_t &= 0.5e^{0.5t} X_T dt + e^{0.5t} dX_t \\ &= 0.5e^{0.5t} X_t dt + e^{0.5}(2dW_t - (1/2)X_t dt) \\ &= 2e^{0.5t} dW_t \end{aligned}$$

(and so)

$$Y_t = Y_0 + \int_0^T 2e^{0.5t} dW_t$$

Thus, $Y_t \sim N(3, \int_0^T 4e^t dt)$ or $Y_t \sim N(3, 4e^T - 4)$. Thus

$$\begin{aligned} Var[Y_7] &= Var[e^{0.5 \cdot 7} X_7] \\ &= e^7 Var[X_7] \end{aligned}$$

(and so)

$$\begin{aligned} Var[X_7] &= e^{-7} Var[Y_7] \\ &= e^{-7}(4e^7 - 4) \\ &= 4 - 4e^{-7} \end{aligned}$$