Midterm Solutions

I. Defintions:

a) We say that a function $f : \mathbb{R} \to \mathbb{R}$ is integrable or L^1 if ...

$$\int_{\mathbb{R}} |f| d\mu < \infty$$

b) Two Borel spaces $(M, \mathcal{A})$ and $(N, \mathcal{B})$ are isomorphic if ...

$$\exists \text{ Borel } f : M \to N \text{ and Borel } g : N \to M$$

$$\text{such that } f \circ g = Id_N \text{ and } g \circ f = Id_M$$

c) Let μ be a measure on $\mathbb{R}$. The cumulative distribution function of μ is the function $F : \mathbb{R} \to [0, 1]$ defined by $F(x) = \ldots$

$$\mu((-\infty, x])$$

d) Let $[a, b]$ be a compact interval. A function $f : [a, b] \to \mathbb{R}$ is said to be of bounded variation if...

$$\text{there exist non-decreasing } g, h : [a, b] \to \mathbb{R} \text{ such that } f = g - h$$

e) If X and Y are two L^2 random variables on a probability space $(\Omega, \mathcal{A}, \mu)$, then the covariance of X and Y is defined by $\mathrm{Cov}[X, Y]$...

$$\frac{Var(X + Y) - Var(X) - Var(Y)}{2}$$

II. True or False:

a) If f is a continuous function on a compact interval $[a, b]$, then the Riemann and Lebesgue integrals of f of $[a, b]$ are equal.

True

b) Any integrable random variable is square integrable.

False

c) Let X be any random variable on a probability space $(M, \mathcal{B}, \mu)$ and let $\mathcal{A}$ be a σ-subalgebra of $\mathcal{B}$. Let Y represent $E[X|\mathcal{A}]$. Then Y is $\mathcal{A}$-measureable.

True

d) Let $f : \mathbb{R} \to \mathbb{R}$ be a measureable function and let μ be a measure on $\mathbb{R}$. Let C be a closed subset of $\mathbb{R}$ and assume that f is supported on C. Then $f\mu$ is supported on C (i.e. concentrated on C).

True

e) If two probability measures on $\mathbb{R}$ have the same Fourier transform, then they are equal.

True

III. Computations

1. Let λ denote Lebesgue measure on $\mathbb{R}$. Find an example of a sequence of integrable functions $f_1, f_2, f_3, \ldots : \mathbb{R} \to \mathbb{R}$ such that BOTH:
(i) for all $x \in \mathbb{R} \quad \lim_{n\to\infty} f_n(x) = 0$; AND
(ii) for all integers $n \geq 1, \quad \int_{-\infty}^{\infty} f_n d\lambda = 1$.

One possible answer is

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [n, n+1] \\ 0 & \text{if } x \notin [n, n+1] \end{cases}$$

2. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$. Let λ denote Lebesgue measure on $\mathbb{R}$ and let $\mu := h\lambda$. Compute $\int_{-\infty}^{\infty} x^6 + 2x^4 + 5x^3 d\mu(x)$.

$$\begin{aligned}
\int_{-\infty}^{\infty} x^6 + 2x^4 + 5x^3 d\mu(x) &= \int_{-\infty}^{\infty} x^6 + 2x^4 + 5x^3 h d\lambda(x) \\
&= \int_{-\infty}^{\infty} x^6 + 2x^4 + 5x^3 \frac{e^{-x^2/2}}{\sqrt{2\pi}} d\lambda(x) \\
&= \int_{-\infty}^{\infty} x^6 \frac{e^{-x^2/2}}{\sqrt{2\pi}} d\lambda(x) + \int_{-\infty}^{\infty} 2x^4 \frac{e^{-x^2/2}}{\sqrt{2\pi}} d\lambda(x) + \int_{-\infty}^{\infty} 5x^3 \frac{e^{-x^2/2}}{\sqrt{2\pi}} d\lambda(x) \\
&= 5 \cdot 3 \cdot 1 + 2 \cdot 3 \cdot 1 + 5 \cdot 0 \\
&= 15 + 6 \\
&= 21
\end{aligned}$$

3. Let Z be a standard normal random variable. Find the variance of e^{2Z+3}.

$$
\begin{aligned}
E[e^{2Z+3}] &= \int_{-\infty}^{\infty} e^{2x+3}\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx \\
&= \frac{1}{\sqrt{2\pi}}e^3\int_{\infty}^{\infty} e^{2x-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}e^3\int_{\infty}^{\infty} e^{-1/2(x-2)^2}e^2dx \\
&= e^5 \\
E[(e^{2Z+3})^2] &= E[e^{4Z+6}] \\
&= \frac{1}{\sqrt{2\pi}}e^6\int_{\infty}^{\infty} e^{4x-x^2/2}dx \\
&= \frac{1}{\sqrt{2\pi}}e^6\int_{\infty}^{\infty} e^{-1/2(x-4)^2}e^8dx \\
&= e^{14}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Var[e^{2Z+3}] &= E[(e^{2Z+3})^2] - (E[e^{2Z+3}])^2 \\
&= e^{14} - e^{10}
\end{aligned}
$$

4. Let $\Omega := [0,1]\times[0,1]$ with its standard σ-algebra $\mathcal{B}$. Let λ_1 be Lebesgue measure on $[0,1]$ and let $\mu := \lambda_1\times\lambda_1$ be the product measure on Ω, so $(\Omega,\mathcal{B},\mu)$ is a probability space. Define random variables U, V on Ω by $U(x,y) = x$ and $V(x,y) = 3x^5 + 4y^6$. Find a random variable W on Ω such that W represents $E[V|U]$.

$$
\begin{aligned}
W &= E[V|U] \\
&= \int_0^1 3x^5 + 4y^6 dy \\
&= (3x^5y + \frac{4}{7}y^7)|_0^1 \\
&= 3x^5 + \frac{4}{7}
\end{aligned}
$$

5. a) Let $I := (1,2)$ be the open interval from 1 to 2. Compute $\delta_2(I)$.

$$\delta_2(I) = 0$$

because $2 \notin (1,2)$.

b) Let $J := [1,2]$ be the closed interval from 1 to 2. Compute $\delta_2(J)$.

$$\delta_2(J) = 1$$

because $2 \in [1,2]$.

c) Compute $\int_{-\infty}^{\infty} x^3 - 2x^2 + 5 d\delta_2(x)$.

$$\begin{aligned}\int_{-\infty}^{\infty} x^3 - 2x^2 + 5 d\delta_2(x) &= 2^3 - 2 \cdot 2^2 + 5 \\ &= 5\end{aligned}$$

d) Find the Fourier transform of δ_2. (Following the definition used in this class, your answer should be an expression of t.)

$$\begin{aligned}\mathcal{F} &= \int_{-\infty}^{\infty} e^{-itx} d\delta_2(x) \\ &= e^{-it2}\end{aligned}$$

6. Let Z be a standard normal random variable.
a) Find the CDF of the distribution of e^{2Z}. Your final answer may involve Φ (the CDF of Z).

$$\begin{aligned}CDF_{\delta_{e^{2Z}}} &= Pr[e^{2Z} < x] \\ &= Pr[Z < \frac{\ln x}{2}] \\ &= \Phi\left(\frac{\ln x}{2}\right)\end{aligned}$$

b) Find a PDF of the distribution of e^{2Z}. Your final answer should not involve Φ or Φ'.

$$\begin{aligned}PDF_{\delta_{e^{2Z}}} &= \frac{d}{dx} CDF_{\delta_{e^{2Z}}} \\ &= \Phi'\left(\frac{\ln x}{2}\right) \cdot \frac{d}{dx}(\frac{\ln x}{2}) \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{((\ln x)/2)^2}{2}} \frac{1}{2x}\end{aligned}$$