Quantum groups are, roughly speaking, deformations of Lie groups (or more generally, reductive algebraic gps.)

Not assuming knowledge of reductive alg. gps / Lie gps, but we hope this gives you motivation / appreciation for them.

Proto-example: $k[X,Y]$: poly. alg. in 2 vars / field k. alg. gen. by X, Y with $XY = YX$.

now $k_q[X,Y]$ has $qXY = YX$

Mention def'n of rep'n here. $\rho: G \to GL(V)$ autom. of V. gps measure symmetry.

Simple example – get plenty of interesting information from this one example, and a few others.

$$SL(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \begin{array}{l} ad-bc=1 \\ a,b,c,d \in \mathbb{C} \end{array} \right\}$$

"special linear gp" $\subseteq GL(2,\mathbb{C})$: invertible 2×2 comp. mats.

Example of PDE invariant under gp action. It's solns are contained in irreps of G.

Lie gp – group + manifold, and gp operation (matrix mult.) behaves well in diff. topology of manifold.

"smooth" $\longrightarrow$

Associated to Lie gp. is Lie algebra – related by exponential map.

(one parameter subgps contained in G)

In this case $X \in Mat_2(\mathbb{C})$ then $\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$

Lie$(SL(2,\mathbb{C}))$ denoted $\mathfrak{sl}(2,\mathbb{C})$

$=$ 2×2 trace 0 complex matrices. , has $[\ ,\ ]$: $[X,Y] = XY - YX$.

As a vector space, spanned by $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

presentation for $\mathfrak{sl}(2,\mathbb{C}) = \left\langle E, F, H \,\middle|\, \begin{array}{l} [H,E] = 2E \\ [H,F] = -2F \\ [E,F] = H \end{array} \right\rangle$

Satisfies basic properties: anti-comm + Jacobi identity

understand this from analytic point of view.

"left-invariant vector fields"

smoothly varying assignment of points to elts of tangent space.

map: rep$(G) \longrightarrow$ rep$(\mathfrak{g} = \text{Lie}(G))$ so $\Pi(e^X) = e^{\pi(X)}$

$\Pi \longmapsto \pi$

$\pi(X) = \frac{d}{dt}\left(\Pi(e^{tX})\right)\Big|_{t=0}$

~~[illegible]~~ what about reverse direction?

Lie algebra rep'ns will have a simple structure, so desirable to have map in opposite direction.

Turns out there is a map, and obstruction to being a bijection is a topological one.

if G is connected, simply connected then this map is 1-1 correspondence.

rep'n theory of $\mathfrak{g} = \mathrm{Lie}(G)$ is captured in the universal enveloping algebra - quotient of tensor algebra

direct sum

$\coprod_i \mathfrak{g}^{\otimes i}$: tensor alg. $/$ $[x,y] - (x \otimes y - y \otimes x)$

contains all rep'ns of Lie algebra. (categories of rep'ns are same)

has center that acts by scalars on irreducibles.

$$\mathcal{U}(\mathfrak{sl}_2) = \langle e, f, h \mid \begin{array}{l} he - eh = 2e \\ hf - fh = -2f \\ ef - fe = h \end{array} \rangle$$

Note: e^2 not matrix mult.

$E^2 = 0$ as matrices.

presentation for $\mathcal{U}(\mathfrak{g})$ using Cartan matrix of assoc. root system.

"some restrictions apply"

q : field elt

$q \neq 0$, $q^2 \neq 1$

Finally, $$\mathcal{U}_q(\mathfrak{sl}_2) = \langle e, f, k, k^{-1} \mid \begin{array}{l} kk^{-1} = k^{-1}k = 1 \\ kek^{-1} = q^2 e, \; kfk^{-1} = q^{-2} f \\ [e,f] = \dfrac{k - k^{-1}}{q - q^{-1}} \end{array} \rangle$$

Not true that naively setting $q = 1$ recovers $\mathcal{U}(\mathfrak{sl}_2)$.

But morally correct. If know presentation for $\mathcal{U}(\mathfrak{g})$, replace integers n in presentation with $[n]_q$ and obtain $\mathcal{U}_q(\mathfrak{g})$.

Quantum Group.

think $k = \begin{pmatrix} q & \\ & q^{-1} \end{pmatrix}$

Where do these relations come from? Turns out as natural consequence of viewing them as Hopf algebras — $\mathcal{U}(\mathfrak{g})$ has a natural Hopf algebra structure.

- Associative alg. over field, with binary op. $m: A \otimes A \to A$
- coassoc. coalgebra : $\Delta: C \to C \otimes C$.

Hopf algebra H has both — m, Δ, units, counits, behave well together.

+ antipode map on H - like inverse in gp.

Spend a few weeks ~~reviewing~~ introducing Hopf algebras and explaining why construction we just presented is natural.

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This is cleanest algebraic approach to quantum gps - same gloss that Drinfeld initiated and presented in ICM lecture at Berkeley. But as Drinfeld readily acknowledges, this wasn't initial motivation for quantum gps.

Original motivation — mathematical physics.

Solvable lattice models — discuss these next time.

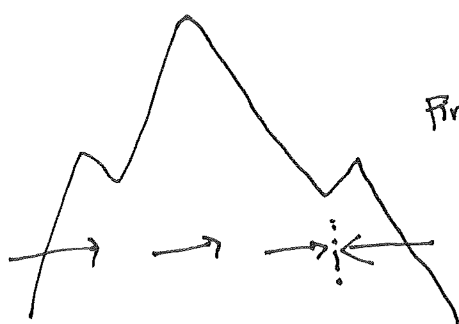
(this is one way my own research touches on aspects of quantum groups)

See Section 7.5 of Chari-Pressley "A Guide to Quantum Groups"

Roughly: Parts I and II of Kassel's book (230+ pages)

Plan :

Next: Dig this way. Hopf algebras. Quantum gp. modules R-matrices



First: Dig this way — Begin from math physics. Lattice models. Arrive at Quantum YBE.

Spend remaining part of semester on special topics.

- especially crystal bases

try to set stage briefly in time remaining -

$\mathfrak{sl}_2$ or $U_q(\mathfrak{sl}_2)$ has generators E, F

where every finite-dim'l module M has basis $v_1, \dots, v_n$ of M

s.t. $E \cdot v_i = \begin{cases} 0 \\ \text{non-zero mult. of some } v_j. \end{cases}$

similarly for $F \cdot v_i$.

Problem for arbitrary Lie algebras $\mathfrak{g}$. now generators $\{E_1, \dots E_k, F_1, \dots, F_k\}$ (indexed by roots in root system)

issue: can't find basis $v_1, \dots, v_n$ for (in general) $\mathfrak{g}$-module M s.t., for all i simultaneously, $E_i \cdot v_j =$ mult. of some basis elt. or 0.

Slogan for what Kashiwara found: "At $q=0$", there is a basis for $U_q(\mathfrak{g})$-modules with this (simultaneous for all roots) property.

make sense of this appropriately. Since $q=0$ not allowed in our description.

This basis has many beautiful properties, explore these.