Last class, we reviewed $\mathcal{U}(\mathfrak{sl}_2)$ repn theory.

At end, we were discussing Hopf module-algebras A:

① - H-module structure on A as vector space

② - $\mu : A \otimes A \to A$, $\eta : k \to A$ are H-module maps

i.e. $\mu(\Delta(x) \cdot (a \otimes b)) = x \cdot (ab) \quad \forall\ x \in H,\ a, b \in A$

$$x \cdot 1 = \varepsilon(x) 1$$

Example: $H = kG$ acting on itself by conjugation. (not left-mult.)

lacks compatibility w/ counit map.

Proposition: Every Hopf alg. acts on itself as a module-algebra via:

$$h \cdot g = \underbrace{\sum h_{(1)} g S h_{(2)}}_{m((\mathrm{id} \otimes S)\Delta h (g \otimes 1))}$$

(Majid, Prop 2.7)

(non-trivial) only interesting when H is not commutative.

So for kG, $\Delta h = h \otimes h$

$S(h) = h^{-1}$

so $h \cdot g \stackrel{\text{def}}{=} h g h^{-1}$ ✓.

Example 2: $\mathcal{U}(\mathfrak{g})$: if $h \in \mathfrak{g}$, then $\Delta h = h \otimes 1 + 1 \otimes h$

$S(1) = 1$, $S(h) = -h$

so $h \cdot g = m(hg \otimes 1) + m(1 \cdot g \otimes -h) = hg - gh$. "adjoint action"

Lemma: For any Lie algebra L, A is a module algebra over $\mathcal{U}(L)$ if and only if A has an L-module structure in which elements act by derivations.

(Recall that L-modules $\longleftrightarrow$ $\mathcal{U}(L)$-modules)

$D: A \to A$. map on algebra acting by Leibniz rule:

$D(ab) = a D(b) + D(a) \cdot b$

pf: $(\Rightarrow)$ Given $x \in L$, then $\Delta(x) = x \otimes 1 + 1 \otimes x$.

Want $m: A \otimes A \to A$ to be an H-module morphism, with $\mathcal{U}(L)$ action ($H = \mathcal{U}(L)$) on $A \otimes A$ given by $\Delta(x) \cdot (a \otimes b)$. So condition required is

$$(*) \qquad \underbrace{x \cdot (m(a,b))}_{\text{Sometimes: } x\cdot(ab)} = \sum_i \underbrace{(x^i_{(1)} a)}_{} (x^i_{(2)} b)$$

Sweedler notation. $m((-) \otimes (\ ))$

Given $\Delta(x)$ as above, then $(*)$ becomes:

$$x(ab) = x(a) b + a\, x(b). \quad \checkmark$$

$(\Leftarrow)$ Follows from fact that we can check mult. is a morphism by checking it on generators. (some calculation about multiplicativity).

Reduction thm. — If A: H-module satisfying unit axiom and mult. axiom on generators. (axiom 2 holding under multiplicativity)

Theorem: Let sl_2 act on polynomials $p \in k[x,y]$: poly. alg. by

$$X \cdot P = x \frac{\partial P}{\partial y}, \quad Y \cdot P = y \cdot \frac{\partial P}{\partial x}, \quad H \cdot P = x \frac{\partial P}{\partial x} - y \frac{\partial P}{\partial y}.$$

Then $k[x,y]$ is a module algebra over $\mathcal{U}(sl_2)$ and submodule $k[x,y]_n$ of homogeneous polys. of degree n is isomorphic to $V(n)$: simple module.

pf of theorem: (a) Check that these formulas define action of sl_2 (compatible with relations)

e.g. $[X, Y] \cdot P = H \cdot P$, etc...

(b) Get algebra action for free, by lemma, since action defined as derivations.

(c) Note the polynomial $P(x,y) = x^n$ is a highest wt. vector of wt n.

determine rest of wt. vectors by applying (suitably normalized powers of Y^p)

get monomials (up to consts.) $x^{n-p} y^p$, $p \le n$, 0 if $p > n$.

which indeed generate $k[x,y]_n \cong V(n)$. //

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idea of "H-module algebra" is interesting. Of course, as algebra alone, we have natural actions - like an algebra acting on itself by right- or left- multiplication.

Here we have "enhanced" action in that it is compatible with bialgebra module structure on $A \otimes A$, k.

Proposition: Every Hopf algebra acts on itself as H-module algebra

via: $$h \cdot g = \sum h_{(1)} g \, S h_{(2)} \qquad \underbrace{m[(id \otimes S)\, \Delta h \, (g \otimes 1)]}$$

η: old axiom about Hopf alg:

$$m \circ id \otimes S \circ \Delta = \varepsilon \circ \eta$$

counit, unit

$$H \xrightarrow{\varepsilon} k \xrightarrow{\eta} H$$

but not quite the same ...

Examples: ① kG: then $\Delta h = h \otimes h$ and $S(h) = h^{-1}$

so $h \cdot g =$ conjugation: hgh^{-1}.

② $\mathcal{U}(\mathfrak{g})$: if $h \in \mathfrak{g}$, then $\Delta h = h \otimes 1 + 1 \otimes h$ and $S(1) = 1$ $S(h) = -h$.

so $h \cdot g = m(hg \otimes 1) + m(1 \cdot g \otimes -h) = hg - gh$ ← "adjoint action"

Back to $U_q(\mathfrak{sl}_2) = \langle E, F, K, K^{-1} \mid KK^{-1} = K^{-1}K = 1$

$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F$

$[E,F] = \dfrac{K - K^{-1}}{q - q^{-1}} \rangle$

<u>Lemma</u>: $\exists$ unique autom. ω of $U_q(\mathfrak{sl}_2)$ with $\omega(E) = F$, $\omega(F) = E$, $\omega(K) = K^{-1}$, $\omega(K^{-1}) = K$ such that $\omega^2 = 1$.

unnecessary to specify. determined by $KK^{-1} = 1$.

<u>pf</u>: ω defined on generators, and $\omega^2 = 1$ on generators,

so ω unique if it exists with $\omega^2 = 1$ everywhere.

For existence, just short check that above is compatible with rel'ns. //

[Similarly, there's a unique antiautom. τ defined on generators by $\tau(E) = E$, $\tau(F) = F$, $\tau(K) = K^{-1}$ with $\tau^2 = 1$.]

These maps are useful for reducing our workload, for example...

<u>Lemma</u>: $(m \geq 0)$

$$[E, F^m] = [m]_q F^{m-1} \frac{q^{-(m-1)}K - q^{m-1}K^{-1}}{q - q^{-1}}$$

$$[E^m, F] = [m]_q \cdot E^{m-1} \frac{q^{m-1}K - q^{-(m-1)}K^{-1}}{q - q^{-1}}.$$

where $[m]_q := \dfrac{q^n - q^{-n}}{q - q^{-1}}$

<u>pf</u>: by induction, using $[E, F^m] = [E, F^{m-1}]F + F^{m-1}[E,F]$

then get second relation from the first by applying ω.

Proposition: $\{ E^i F^j K^l \}_{i,j \geq 0,\, l \in \mathbb{Z}}$ are a basis for $U_q(sl_2)$.

(reorder them if you prefer - put F's first, etc.)

pf: Same as for $U_q(b_+)$ - show span since span of set is stable under left mult. by generators. Easy for K, K^{-1}, since commutation relations are easy. For E, F, make us of previous lemma.

For linear independence, make map $U_q \hookrightarrow \mathrm{End}_k(k[x,y,z,z^{-1}])$

this ring has same basis: $\{x^i y^j z^l\}$

Show endoms. are linearly indep. by evaluation of $e^i f^j k^l$ at 1.

$E \mapsto e : x^i y^j z^l \mapsto x^{i+1} y^j z^l$

etc.

(one for F is messier, have to do commutation relation)

Corollary: $U_q(sl_2)$ has no zero divisors

pf: Write this basis in order $F^s K^t E^r$, consider subalgebra $U_0 \subset U$ generated by K, K^{-1} $\cong k[z, z^{-1}]$

then any elt in U_q is expressible as linear combination of the form

$F^s \cdot h \cdot E^r$ with $h \in U_0$, $r, s \geq 0$.

with "leading terms" (biggest monomial with (s,r)) and remaining monomials with pairs (s', r') with either $s' < s$ OR $s' = s$ and $r' < r$.

Finally, show leading terms behave well under multiplication,

$m : (s,r), (p,m) \rightsquigarrow (s+p, r+m)$

so if $u, v \neq 0$, then $uv \neq 0$.