Again for $U_q(\mathfrak{sl}_2)$, have notion of weight vector (evector for the action of K)

Let $M_\lambda = \{ m \in M \mid K \cdot m = \lambda m \}$ (λ-eigenspace)

then relations for $U_q(\mathfrak{sl}_2)$ imply, for any λ,

$$E \cdot M_\lambda \subseteq M_{q^2\lambda} \qquad F \cdot M_\lambda \subseteq M_{q^{-2}\lambda}.$$

(and sum of the M_λ is direct). If we form $\underbrace{\bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}}_{\text{finite sum if } q \text{ is rt. of unity.}}$,

this is submodule of any $U_q(\mathfrak{sl}_2)$-module M.

If $M_\lambda \neq 0$ for some λ and M is simple,

then $M = \bigoplus_{n \in \mathbb{Z}} M_{q^{2n}\lambda}$ (generated by any non-zero vector)

Trouble from prior proof for $U(\mathfrak{sl}_2)$: Assumed k: field is algebraically closed to guarantee non-zero M_λ.

Want to take a different approach applicable to case where q ~~not~~ rt. of unity, $\mathrm{char}(k) \neq 2$.

Prop: Suppose q is not a root of unity. Let M be finite dim'l $U_q(\mathfrak{sl}_2)$-module.

$\exists$ integers r, s s.t. $E^r \cdot M = 0 = F^s \cdot M$ (meaning these powers annihilate every $m \in M$)

pf: Let $M_{(f)} = \{ m \in M \mid f(K)^n \cdot m = 0 \quad n \gg 0 \}$ f: irreducible one var. poly. coeffs. in k.

Since M finite dim'l, then M is direct sum of distinct

$M_{(f)}$, where if $M_{(f)}, M_{(g)} \neq 0$, then $M_{(f)} = M_{(g)}$ iff f, g differ by a non-zero constant factor.

Given $f \in k[X]$ with $M_{(f)} \neq 0$, let $f_i(X) := f(q^{+i}X)$

(note f_i is again irred. if f is, since it is the image of f under automorphism of $k[X]$ def'd by $X \mapsto q^i X$.)

Recall from original rel'ns, $EK = q^{-2}KE$, so

$$E \cdot f(K) = f(q^{-2}K) \cdot E \quad \text{and by induction on } r:$$

$$E^r \cdot f(K) = f_{-2r}(K)\, E^r \qquad (*)$$

this implies $\quad E^r \cdot M_{(f)} \subseteq M_{(f_{-2r})} \quad \forall\, r \geq 0$

(if $x = E^r \cdot m$, with $m \in M_{(f)}$, then $f_{-2r}(K)^n E^r m \overset{(*)}{=} E^r f(K)^n m = 0$ for $n \gg 0$ since $m \in M_{(f)}$)

So if we can show, for every f,

$M_{(f_{-2r})} = 0$ for some $r > 0$, then done

(as every $m \in M$ belongs to some $M_{(f)}$, and take max over all such r.)

Suppose $M_{(f_{-2r})} \neq 0 \;\; \forall\, r > 0$. Then eventually, must have some r, s with

$$M_{(f_{-2r})} = M_{(f_{-2s})}$$

$\Rightarrow$ f_{-2r} and f_{-2s} proportional (by elt. of k)

but $f(q^{-2r}x)$, $f(q^{-2s}x)$ have same constant term

(only irreducible with constant term 0 is X, but $M_{(X)} = (0)$ since K is invertible, so $K^n \cdot m \neq 0$ $\forall\, m$.)

on other hand, if f has degree n, then leading terms differ (for f_{-2r}, f_{-2s})

by $q^{2(s-r)n} \neq 1$ since q not a root of unity. ↯.

(similar pf. to show sufficiently large power of F annihilates any such M.)

Corollary: (q not rt. of unity, char$(k) \neq 2$) Let M be fin. dim'l $U_q(\mathfrak{sl}_2)$ module. Then M is a direct sum of its weight spaces. All weights have form $\pm q^a$ with $a \in \mathbb{Z}$.

pf: Endom. of fin. dim'l v.s. is diagonalizable if and only if its minimal polynomial splits into linear factors, each occurring with mult. 1.

monic poly. of least degree annihilating endom.

(divides characteristic poly.)

Investigate min. poly. for K acting on M.

By previous result, $\exists$ integer $s > 0$ s.t. $F^s M = 0$. Set

$$h_r^{(s)} = \prod_{j=-(r-1)}^{r-1} \frac{Kq^{r-s+j} - K^{-1}q^{-(r-s+j)}}{q - q^{-1}}$$

claim: For $0 \leq r \leq s$, $F^{s-r} h_r \cdot M = 0$. (induction pf.)

In case $r = s$, just get $h_s \cdot M = 0$. Clean up non-zero constant factors mult. by power of K

then

$$0 = \left(\prod_{j=-(s-1)}^{s-1} (K^2 - q^{-2j}) \right) M = \left(\prod_{j=-(s-1)}^{s-1} (K - q^{-j})(K + q^{-j}) \right) M$$

And hence the minimal polynomial of K divides the associated polynomial to the above (replace K with variable x).

So we're done! Because this polynomial splits into linear factors with distinct roots, so the min. poly. must also.

(if you were worrying that maybe $-q^a = q^b$ some a, b, note this implies $q^{b-a} = -1$ so $q^{2(b-a)} = 1$ ↯ q not a root of unity.)

— END OF LECTURE 17 —

Now we know that fin-dim'l modules are entirely comprised of wt. spaces.

let's explore highest weights: $m \in M$, $Em = 0$, $Km = \lambda m$.

There's a natural "universal" module with this property:

$$M(\lambda) := \mathcal{U}_q \big/ \mathcal{U}_q E + \mathcal{U}_q (K - \lambda) .$$

Any M with h.w. vector λ m of wt. has a unique homom. $M(\lambda) \longrightarrow M$

$[\cdot]$ for coset rep classes. $[1] \longrightarrow m$

This $M(\lambda)$ has basis $[F^i]_{i \geq 0}$

by PBW theorem, call these $m_0, m_1, m_2, \ldots$

then our relations imply:

$$F \cdot m_i = m_{i+1}$$

(infinite dim'l module)

$$E \cdot m_i = \begin{cases} 0 & \text{if } i = 0 \\ [i]_q \dfrac{\lambda q^{1-i} - \lambda^{-1} q^{i-1}}{q - q^{-1}} m_{i-1} & \text{otherwise.} \end{cases}$$

$$K \cdot m_i = \lambda \cdot q^{-2i} m_i$$

⟵ already make some observations about λ being integer power of q or not