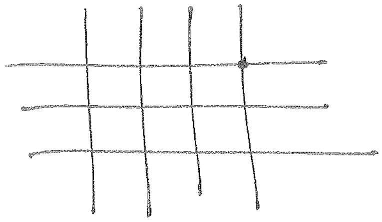
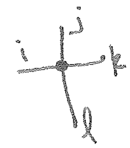
Quantum YBEs : Statistical mechanics - infer macro behavior from microscopic interactions. ①

Begin with integrable lattice models - model of local interactions



each crossing has vertex



"atoms"

with adjacent edges ~~describing~~ energy from interactions with nearest neighbor atoms

"bonds"

E.g. 6-vertex model :

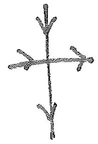
$i, j, k, l \in \{ \rightarrow, \leftarrow \}$

$E_{i,j}^{k,l} = 0$ unless 2 arrows in, 2 arrows out.

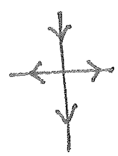
Given i, j, k, l in some finite set of possibilities,

write down energy $E_{i,j}^{k,l}$ at the atom.

So



ok, not



(6 such configurations)

still need to tell you about $E_{i,j}^{k,l}$ at these ...

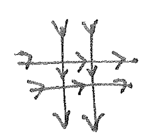
this nearest neighbor interaction is reasonable in many systems. Inert gases : Intermolec. force goes by r^{-7} r : dist. btw. molecules

Goal : Infer global behavior for local interactions.

e.g. total energy on finite rectangle $\displaystyle\sum_{v: \text{ vertices in rect.}} E_{i,j}^{k,l}(v)$

Typically, study finite rectangles with fixed boundary conditions, investigate various limits as # rows, # columns $\rightarrow \infty$.

(Do small example)



probability that atoms arrange in given config. is inversely proportional to energy.

More precisely : $e^{-\beta \cdot \text{Energy (state)}}$ where $\beta = 1/kT$

derived from ideal gas law $pV = nRT$

k : Boltzmann's constant

T : temperature.

Except we need to normalize by total probability.

$$Z := \sum_{\text{states}} e^{-\beta \, \text{Energy}(\text{state})}$$

"partition function" of system.

$$= \sum_{\text{states}} \prod_{\substack{v:\text{ vertices} \\ \text{in state}}} \underbrace{e^{-\beta E_{ij}^{k,l}(v)}}_{R_{ij}^{k,l}(v)}$$

"Boltzmann weight" of vertex v.

and then given physical quantity Q,

its average value is then =

$$\frac{\sum_{\text{states}} Q \cdot e^{-\beta \cdot \text{Energy}(\text{state})}}{\underbrace{\sum_{\text{States}} e^{-\beta \cdot \text{Energy}(\text{state})}}_{Z}}$$

e.g. if $Q = \text{Energy}(\text{state})$

$$\underset{\langle E \rangle}{\text{average energy}} = k \cdot T^2 \frac{\partial}{\partial T} \ln Z$$

check this!

upshot: Want to explicitly evaluate Z. If we can, say model is "exactly solvable"

(a little bit cheap, because what does "explicitly evaluate" mean? Match some artificial list of functions with a name?)

Example: 6-vertex model Boltzmann weights. Famous 2 parameter family.

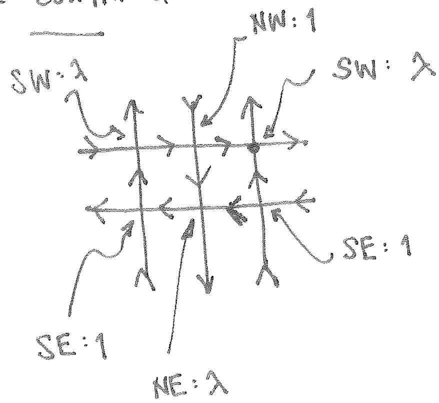
Compass directions for in arrows

SE NW NE SW NS EW

Boltzmann weights: $\underbrace{\text{SE} \quad \text{NW}}_{1}$ $\quad \underbrace{\text{NE} \quad \text{SW}}_{\lambda}$ (complex parameter) $\quad \underbrace{\text{NS}}_{1-q\lambda} \quad \underbrace{\text{EW}}_{1-q^{-1}\lambda}$

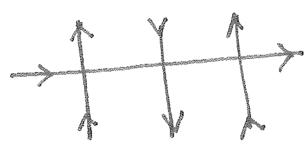
(q = $\neq 0$ cx-parameter)

Example continued:

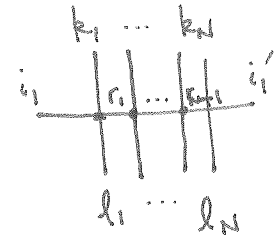


total wt: λ^3.

Plan: Build up inductively from 1-row systems. Then have:



or



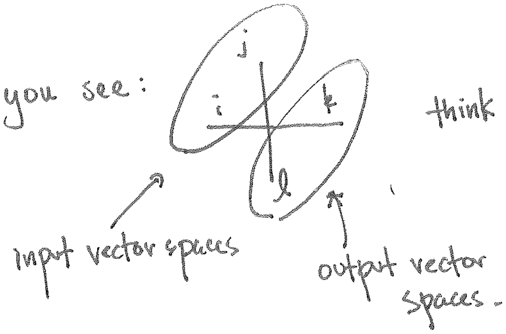
Let $T^{i_1'; \underline{k}}_{i_1; \underline{l}}$: partition function of one-row system.

$$\text{then} \quad T = \sum_{r_1, \ldots, r_{N-1}} R^{r_1, k_1}_{i_1, l_1} \cdots R^{i_1', k_N}_{r_{N-1}, l_N}$$

clever algebraic bookkeeping device — if decorations have M choices, bookkeep indices with M-dimensional vector space V with basis $\{v_1, \ldots, v_M\}$. Then define $R \in \mathrm{End}(V \otimes V)$ by specifying it on a basis:

$$R(v_i \otimes v_j) = \sum_{k,l} R^{k,l}_{i,j} (v_k \otimes v_l).$$

So when you see:

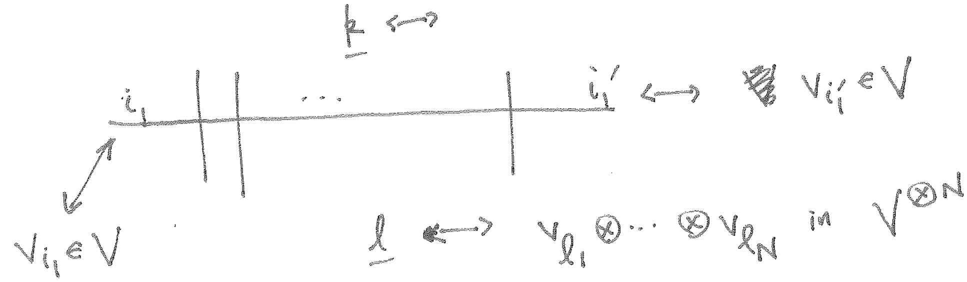


think: Boltzmann weight $R^{k,l}_{i,j}$ is a piece of endom. with

$$v_i \otimes v_j \longmapsto \cdots + R^{k,l}_{i,j} v_k \otimes v_l + \cdots$$

Similarly view T: one row partition function as elt. of $\mathrm{End}(V \otimes \underbrace{V^{\otimes N}}_{N\text{-fold tensor}})$ where N is the # of columns.

so for T:



claim: $T = R_{01} R_{02} \cdots R_{0N}$ where R_{0j} means acting on

$V \otimes (V \otimes \cdot V \otimes V)$

j^{th} factor of N factors

R acts on this copy of $V \otimes V$. and as identity on all other copies.

Example: 6-vertex model.

V : 2 dim'l vector space $= \langle \uparrow, \downarrow \rangle$
in out

$V \otimes V$ is 4 dim'l.

$\mathrm{End}(V \otimes V)$ is encoded as 4×4 matrix. so R is such.

$$T \in \mathrm{End}(\underbrace{V}_{2\text{ dim'l}} \otimes (\underbrace{V^{\otimes N}}_{2^N \text{ dim'l}}))$$

We make matrix $R \in \mathrm{End}(V \otimes V)$

whose entries are then indexed by

need to explain choice of orientation.

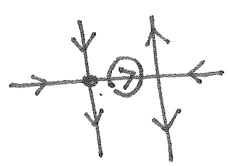
$$\begin{array}{c|c} & \uparrow,\uparrow \quad \uparrow,\downarrow \quad \downarrow,\uparrow \quad \downarrow,\downarrow \leftarrow \text{entries of } V\otimes V \\ \hline \uparrow,\uparrow & \\ \vdots & \begin{pmatrix} & & & \\ & & & \end{pmatrix} \\ \downarrow,\downarrow & \end{array}$$

Chari–Presley: v_1 : up or left
v_2 : down or right

Weights in earlier example:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 - q\lambda & 0 \\ 0 & 1 - q^{-1}\lambda & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

SE
EW
NW



matrix entry in $\mathrm{End}(V \otimes (V \otimes V))$

$$T \circlearrowright V \otimes (\underbrace{V \otimes V}_{8\text{ dim'l}})$$

8×8 matrix

$$R_{01} : \begin{pmatrix} \boxed{R} & 0 \\ 0 & \boxed{R} \end{pmatrix}, \quad R_{02} :$$

$$
\begin{array}{c|cccccccc}
 & +++ & +-+ & -++ & --+ & ++- & +-- & -+- & --- \\
\hline
+++ & 1 & & & & & & & \\
+-+ & & 1 & & 1-q\lambda & & & & \\
-++ & & & \lambda & & 1-q^{-1}\lambda & & & \\
--+ & & & & \lambda & & 1-q^{-1}\lambda & & \\
++- & & & 1-q\lambda & & \lambda & & & \\
+-- & & & & & & \lambda & & \\
-+- & & & & & & & 1 & \\
--- & & & & & & & & 1
\end{array}
$$

column 6 of R_{02}

We want the element of form

$--+, \; +--$

$\big(\underset{\text{row}}{4} \;,\; \underset{\text{column}}{6} \big)$

$$\begin{vmatrix} 0 \\ 0 \\ 0 \\ 1-q^{-1}\lambda \\ 0 \\ \lambda \\ 0 \\ 0 \end{vmatrix}$$

✓. $\boxed{1-q^{-1}\lambda}$

in R_{01} row 4: $\underline{0\ 0\ 0\ 1}\ \big|\ 0\ 0\ 0\ 0$ in $R_{01}R_{02}$.

Hope this is equal to entry of T at corresponding position.

For example, we could require left-boundary conditions equal to the right-boundary conditions and then, summing over all boundary conditions ("toroidal boundary cond.")

then the partition function (over all such choices) is:

Prop. 7.5.1 in Chari-Pressley: $\quad Z = \operatorname{trace}_{V^{\otimes N}} \big(\operatorname{trace}_V (T)\big)^M$

N: columns. M: rows