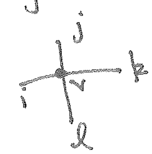
Last time, we were trying to understand "partition function"

$$Z = \sum_{\substack{\text{admissible}\\ \text{states}\\ S}} \prod_{\substack{x \in S\\ x:\text{vertex}}} R_{i,j}^{k,l}(x)$$

R: Boltzmann weight for



<u>motivation</u> - Infer macroscopic thermodynamics from R - probability of state according to its energy.

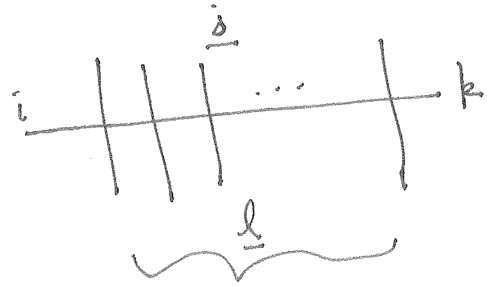
but we can just think of $R_{i,j}^{k,l}(x)$ as abstract function. (produce interesting slew of Z's : special functions from careful choices of function R)

We had transformed this into algebraic context : $V = \langle v_1, \dots, v_m \rangle$ where m : # of possible decorations on edges

define matrix $R \in \mathrm{End}(V \otimes V)$ whose coeffs. are $R_{i,j}^{k,l}$

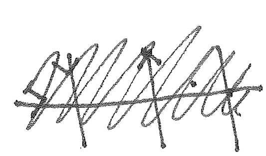
Similarly, matrix $T \in \mathrm{End}(V \otimes (V^{\otimes N}))$ whose coeffs are

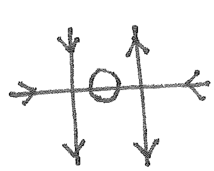
$T_{i,\underline{j}}^{k,\underline{l}}$: the partition function of the system with 1 row, N columns



N columns

In 6-vertex model, might want algebraic manifestation of





system with one internal edge.

in our notation : $T_{\rightarrow, \downarrow\uparrow}^{\leftarrow, \uparrow\downarrow}$

And every other choice of boundary arrows would give a different matrix coeff. in T.

matrix coeff. in $T \in \mathrm{End}(V \otimes (V^{\otimes 2}))$

Claim: $T = R_{01} R_{02} \cdots R_{0N}$ for any R, T corresponding to lattice model.

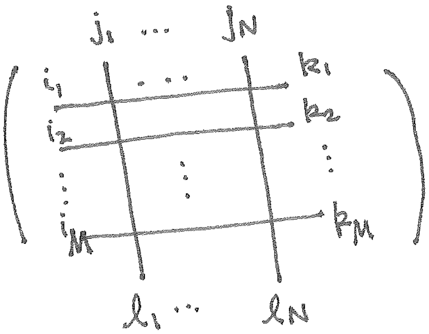
in our example, $T = R_{01} R_{02}$

where $R_{01} \in \mathrm{End}(\underbrace{V \otimes (V^{\otimes 2})}_{V_{(0)} \otimes (V_{(1)} \otimes V_{(2)})})$ means $R \otimes \mathrm{id}$: with R acting on $V_{(0)} \otimes V_{(1)}$ and id. on $V_{(2)}$.

similarly R_{02} means act by id. on $V_{(1)}$ and by R on $V_{(0)} \otimes V_{(2)}$.

so if $\underset{\sim}{R} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1-q\lambda & 0 \\ 0 & 1-q^{-1}\lambda & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, try to describe partition function on previous page.

(see example in prior notes)

Then Z () $= \sum_{\underline{r}^{(M-1)}} \cdots \sum_{\underline{r}^{(1)}} T_{i_1, \underline{j}}^{k_1, \underline{r}^{(1)}} \cdot T_{i_2, \underline{r}^{(1)}}^{k_2, \underline{r}^{(2)}} \cdots T_{i_M, \underline{r}^{(M-1)}}^{k_M, \underline{l}}$

(*)

messy in general to explain/simplify this calculation.

If we use toroidal boundary conditions (view boundary edge as internal edge wrapped around)

In one-row partition function, $Z = \mathrm{trace}_V (T)_{\underline{j}}^{\underline{l}}$

sum of diagonal terms in map $V \otimes V^{\otimes N} \to V \otimes V^{\otimes N}$

so this is still $(\dim V)^N \times (\dim V)^N$ matrix so $\mathrm{trace}_V (T) : V^{\otimes N} \to V^{\otimes N}$

and now the above really is just matrix multiplication. so in multi-row Z we get...

Z in $(*)$ is equal to $\qquad \left((\mathrm{trace}_V(T))^M \right)_{\underline{j}}^{\underline{l}}$

(with horizontal edges having toroidal boundary conditions)

If we also make vertical boundary edges toroidal (so edges in $\underline{l}$, $\underline{j}$ internal)

then we take another trace on $V^{\otimes N}$, get

$$Z = \mathrm{trace}_{V^{\otimes N}} \left((\mathrm{trace}_V (T))^M \right).$$

(Proposition 7.5.1. in Chari-Pressley really should emphasize: "toroidal boundary conds.")

<u>Note</u>: Can choose any other boundary conditions, just get less familiar functionals that pick off a particular matrix coeff. in $\mathrm{End}(V \otimes V^{\otimes N})$.

In particular as $M \to \infty$, $\quad Z \sim K_N^M \quad$ where K_N : largest e-value of $\mathrm{trace}_V(T)$ in $\mathrm{End}(V^{\otimes N})$

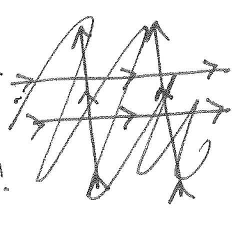
How can we find this e-value?

(so biggest among 2^N e-values assuming they are distinct)

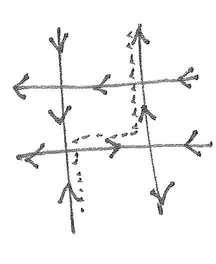
Model dependent. In the context of 6-vertex model with toroidal boundary conditions, then

admissible decorations of lattice $\longleftrightarrow$ 2 in / 2 out arrows at each vertex.

$\longleftrightarrow$ also assign path to, say, up and ~~[illegible]~~ left arrows.



Bad example. Too many paths!



then admissible states correspond to non-crossing paths moving through lattice.

toroidal boundary conditions ensure that paths never end -

old arcade / space invaders geometry - if path exits out right, then

comes back on left.

Want evalues of $\operatorname{trace}_V(T) \in \operatorname{End}(V^{\otimes N})$ ←

Annoying to write $\operatorname{trace}_V(T)$ and understand it as collapsed T. ↑ just write "T" for collapsed version?

cases based on $n :=$ number of up edges along bottom of one row system (coeff. of T).

if $n=0$, boring. piece of T is a 1×1 matrix

with e-value $\lambda = wt(SE)^N + wt(SW)^N$

and on and on. (see Ch. 8 of Baxter's book)

formulas get ever more complicated.

Guess shape of eigenfunctions, find all e-values. (Bethe Ansatz)

Lieb / Sutherland in ~1967 to "solve" 6-vertex model with toroidal bdy. conditions.

So then positions (column index) of up arrows recorded

$$\underbrace{x_1 < x_2 < \cdots < x_n}_{\substack{\parallel \\ X}}$$

if we had e-value λ for T and e-vector g.

$$g(X)\cdot\lambda = \sum_Y T(X,Y)g(Y)$$

Because of toroidal bdy. conditions, ensures break T up into blocks.

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if not toroidal, then what?

6 weights ⟶ symmetric on reversing arrows "field-free" case.

3 weights.

In nice cases: we can diagonalize our transfer matrix T:

$$T = \underbrace{P}_{\substack{\text{matrix} \\ \text{of eigenvectors}}} \underbrace{T_{diag.}}_{\substack{\text{e-values} \\ \text{(distinct)}}} P^{-1}$$

, and find that P is independent of some parameters. call it x_i for T_i : in row i

"transfer matrices commute"

But ~~[illegible]~~ diag. matrices commute, so

$$T(x_1)T(x_2) = T(x_2)T(x_1)$$