Last time, giving first results on partition functions.

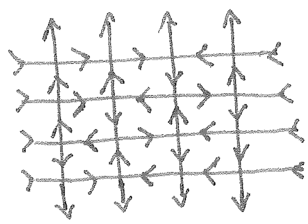
If $M=N$ (square lattice), Boltzmann wts in 6-vertex model all $=1$,

$$\begin{array}{ccc} & \uparrow\uparrow & \\ \rightrightarrows & & \leftleftarrows \\ & \downarrow\downarrow & \end{array}$$

(domain wall bdy. conditions)

then $Z_N = \sum_{S \in \text{admissible state}} 1 = \#$ of ASMs (alternating sign matrices)

bijection:



$$\longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

EW: 1, NS: -1, all else: 0.

Sketch pf. of following theorem (Kuperberg): $Z_N = \dfrac{1!\,4!\,7! \cdots (3N-2)!}{N!\,(N+1)! \cdots (2N-1)!}$

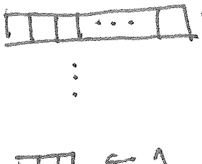
(return to earlier notes pages)

Note: Zeilberger's proof was not bijective. Just showed equinum.

~~Recall~~ Recall Schur polynomial – very special symmetric function (character values of highest weight representations V_λ of $SL_n(\mathbb{C})$)

combinatorial definition in terms of semi-standard Young tableaux (SSYT)

Given partition $\lambda = (\lambda_1, \dots, \lambda_r)$, make diagram of shape λ



← λ_1 boxes

⋮

← λ_r boxes

ex: $\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} = T.$ for $\lambda = (5,3,1)$

SSYT: filling with alphabet $\{1, \dots, r\}$ s.t. weakly increasing in rows, strictly increasing in columns

$wt(T) = z_1^{\#1\text{'s}} \cdots z_r^{\#r\text{'s}}$

in our example $wt(T) = z_1^2 z_2^4 z_3^3$.

Resources: Kuperberg "Another pf. of ASM conjecture"

arxiv: math/9712207

Follow slightly slicker proof (Stroganov / Okada indep.) in

Zinn-Justin "Six-vertex, loop, and tiling models: integrability and combinatorics."

Field-free Boltzmann weights

$a = qx - q^{-1}x^{-1}$ (SW = NE)

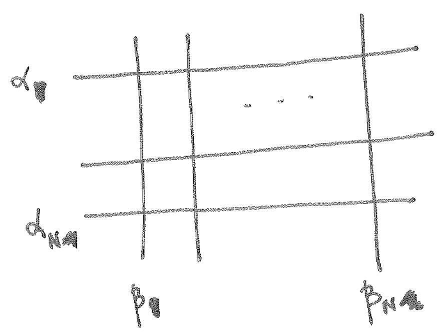
$b = x - x^{-1}$ (SE / NW)

$c = q - q^{-1}$ (NS / EW)

$\Delta = \frac{q + q^{-1}}{2}$. (indep. of parameter x so has YBE).

Label rows and columns as follows:

with $x = \alpha_i / \beta_j$ at each vertex.



Get a partition function

$Z_N(\underline{\alpha}, \underline{\beta})$ with following properties:

- Z_N is symmetric in $\underline{\alpha}, \underline{\beta}$ $(\alpha_i \leftrightarrow \alpha_{i+1})$ / $(\beta_i \leftrightarrow \beta_{i+1})$
- $\alpha_i^{N-1} Z_N$ is degree at most $N-1$ in (α_i^2)
  $\beta_i^{N-1} Z_N$ ——— " ——— in (β_i^2)
- $Z_N(\underline{\alpha}, \underline{\beta}) = (q - q^{-1}) \prod_{i=2}^{N} (q\alpha_1/\alpha_i - q^{-1}\alpha_i/\alpha_1) \prod_{j=2}^{N} (q\beta_j/\alpha_1 - q^{-1}\alpha_1/\beta_j)$
  if $\alpha_1 = \beta_1$: $\cdot Z_N(\alpha_2, \dots, \alpha_N; \beta_2, \dots, \beta_N)$

Now specialize at $q = e^{i\pi/3}$. At this specialization

$Z_N(q^{-1}, \dots, q^{-1}, 1, \dots, 1)$ is, up to an explicit const, # of $N \times N$ ASMs.

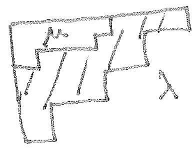
$S_\lambda(\underline{z}) := \sum_{T \in SSYT(\lambda)} \underline{z}^{wt(T)}$.   Rep'n theory: $\rho_\lambda : GL_n(\mathbb{C}) \longrightarrow \mathrm{End}(V)$

$\cup$
$A \ni \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}$
diag mats.

Also skew Schur polys $S_{\lambda/\mu}$ where

$S_\lambda(\underline{z}) = \mathrm{Tr}\left(\rho_\lambda\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_r \end{pmatrix}\right)$

we fill skew tableaux:



Skew shape λ/μ.

$S_{\lambda/\mu}(\underline{z}) := \sum_{T \in SSYT(\lambda/\mu)} \underline{z}^{wt(T)}$

Given partition λ (r parts) $\rightsquigarrow$ partition $\lambda+\rho$ with distinct parts
$\rho = (r-1, r-2, \dots, 1, 0)$

$\rightsquigarrow$ row of arrows : up at columns index matches parts of $\lambda+\rho$.
down else.

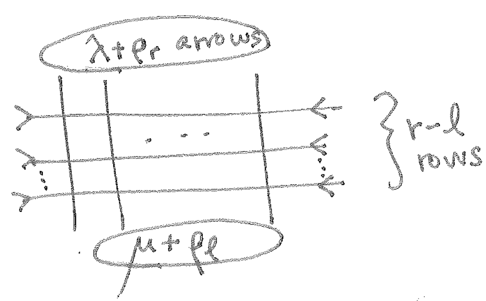
$\lambda = (2,2,0) \rightsquigarrow \lambda+\rho = (4,3,0)$

(bijection between partitions and rows of up/down arrows)

$$\rightsquigarrow \begin{array}{ccccc} \uparrow & \uparrow & \downarrow & \downarrow & \uparrow \\ 4 & 3 & 2 & 1 & 0 \end{array}$$

Given λ, μ, $\mu \le \lambda$, then make lattice

λ : r parts
μ : ℓ parts



(i.e. generic columns, prescribed rows)

$\mu = \phi$, $\lambda = (\underbrace{0, \dots, 0}_{r})$
is Domain Wall Bdy conditions

Thm: (same paper w/ Brubaker-Bump-Friedberg)

2011 - Commun. in Math Physics, $\mu = \phi$ case:

$$Z_{\lambda+\rho} = Z_\rho \cdot S_\lambda\left(\frac{b_2^{(1)}}{a_1^{(1)}}, \dots, \frac{b_2^{(r)}}{a_1^{(r)}}\right)$$

for any wts. with $\Delta^{(i)} = 0 \;\forall i$

More complicated for μ gen.: $S_{\lambda/\mu}$

with $Z_\rho = \prod_{i=1}^{r} a_1^{(i)} c_2^{(i)} \cdot \prod_{i<j} \left(a_1^{(j)} a_2^{(i)} + b_1^{(i)} b_2^{(j)}\right)$