

Last time, defined bialgebra - an algebra (m, η) and a coalgebra (Δ, ε) with compatible defns. (expressed via commutative diagrams) ①
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Example: kG is bialgebra with $m(e_g, e_h) = e_{g*h}$, $\Delta(e_g) = e_g \otimes e_g$

check, for example, that

$$\Delta(m(e_g, e_h)) = m(\Delta(e_g), \Delta(e_h))$$

$$\Delta(e_{g*h}) = m((e_g \otimes e_g) \otimes (e_h \otimes e_h))$$

$$= e_{g*h} \otimes e_{g*h} = e_{g*h} \otimes e_{g*h} \quad \checkmark$$

means $\text{id} \otimes \tau \otimes \text{id}$, then $m \otimes m$.

As some of us discussed after class on

Friday, given any set S , form a coalgebra on vector space $k^{|S|}$ with same definitions. But in case of kG , get bialgebra.

Nevertheless, subset of elts. $x \in H$ with $\Delta(x) = x \otimes x$ will be "group-like elements" important in sequel.

One nice property of bialgebra - Given algebra modules V, W , then $V \otimes W$ has natural $H \otimes H$ -module structure. But if we compose with coproduct $\Delta: H \rightarrow H \otimes H$, we get natural H -module

So you may have seen G -action on $V \otimes W$ given by $g \cdot (v \otimes w) = gv \otimes gw$
(or equivalently kG action)

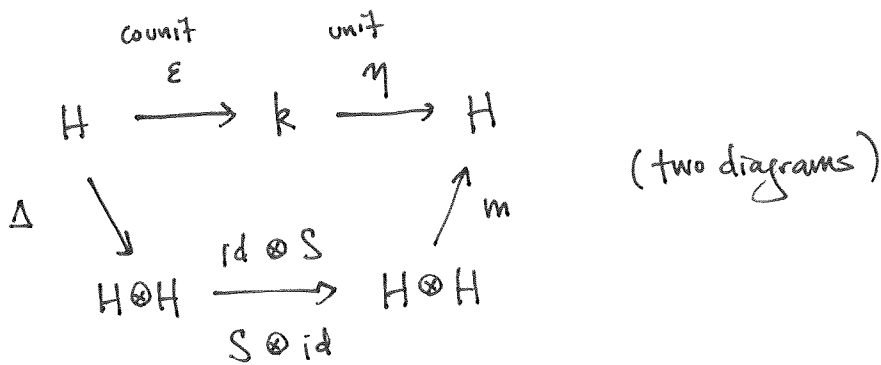
Similarly tensor algebra has action from complicated Δ structure.

now understand that as coproduct in kG
 $\Delta: g \mapsto g \otimes g.$

One further axiom to get from bialgebra to Hopf algebra -

Antipode map $S: H \rightarrow H$ with $m \circ (S \otimes id) \circ \Delta = \eta \circ \epsilon$
 $= m \circ (id \otimes S) \circ \Delta$

or in diagrams:



kG example: $S: e_g \mapsto e_{g^{-1}}$ (indeed, think of antipode as a "quantum gp" version of inversion)

$T(V)$ example: $S: v \mapsto -v$
 in $T'(V)$,
 extend. note - not requiring $S^2 = id$.

Proposition on antipodes S : S is an anti-algebra map $S(hg) = S(g)S(h)$
 and anti-coalgebra map $S(1_H) = 1_H$
 $\forall h, g$.

$$(S \otimes S) \circ \Delta = \tau \circ \Delta \circ S$$

$$\epsilon \circ S = \epsilon$$

Moreover S is unique.

pf: For example, for uniqueness, mimic pf. that gp. inverses are unique.

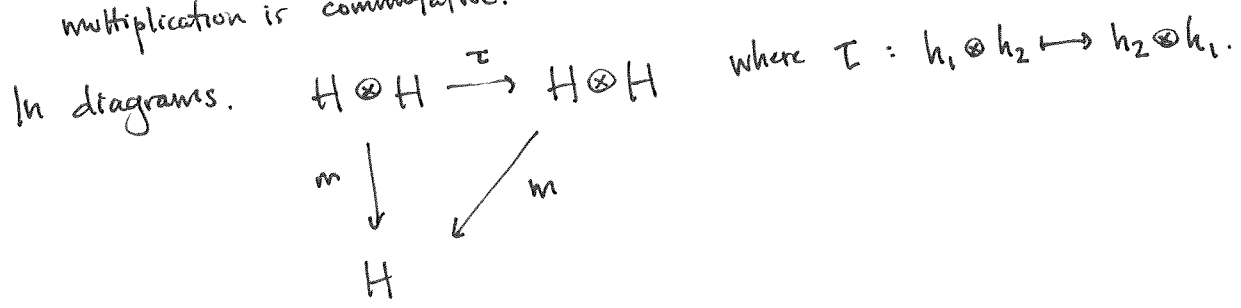
Suppose there were two: S, S'

then $S'(h) = S'(h_{(1)} \epsilon(h_{(2)})) \stackrel{\text{linearity}}{=} S'(h_{(1)}) \epsilon(h_{(2)})$
 $\stackrel{\text{counit axiom}}{=} S'(h_{(1)}) m \circ (id \otimes S) (h_{(2)(1)} \otimes h_{(2)(2)})$
 $\stackrel{\text{antipode axiom}}{=} S'(h_{(1)}) m \circ (id \otimes S) (h_{(2)(1)} \otimes S(h_{(2)(2)}))$

$$\begin{aligned}
 \text{Had} &= S'(h_{(1)}) m(h_{(2)(1)}, Sh_{(2)(2)}) \\
 &= S'(h_{(1)(1)}) h_{(1)(2)} Sh_{(2)} \quad (\text{coassociativity}) \\
 &= \varepsilon(h_{(1)}) Sh_{(2)} \quad (\text{antipode axiom}) \\
 &= S(h) \quad (\text{counit axiom})
 \end{aligned}$$

Coassociativity in Sweedler notation: $c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$

Further adjects: commutativity / cocommutativity.
 multiplication is commutative.



and for cocommutative, as usual, we reverse the arrows:

$$\tau \circ \Delta = \Delta \text{ on } H \rightarrow H \otimes H.$$

Some things are immediately nicer:

Proposition: If H is commutative or cocommutative, then $S^2 = \text{id}$.

Check: $kG, T(V)$ are cocommutative. (eg. $e_g \cdot e_h = e_{gh}$, so kG commutative if G is abelian. check that $T(V)$ is only commutative if $\dim(V) \leq 1$.)

Immediate from def'n - If H is cocommutative then H -module structure on $V \otimes W$ is isomorphic to that of $W \otimes V$.

Another example: \mathfrak{g} : Lie algebra - vector space with a (J) (4)
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 Lie bracket $[\cdot, \cdot]: \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying Jacobi identity + antisymmetry props (if $\text{char}(k) \neq 2$) (A)

then $U(\mathfrak{g})$ is quotient of $T(\mathfrak{g})$: tensor Hopf algebra on vector space \mathfrak{g}
 with relation $g_1 \otimes g_2 - g_2 \otimes g_1 = [g_1, g_2]$
 $(J): [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$
 $(A): [x, y] = -[y, x]$

This gives Hopf algebra structure on $U(\mathfrak{g})$, which remains cocommutative.

(Hopf's initial motivation - unified treatment for groups, Lie algebras particularly their cohomologies)

Majid's comments: - Any theorem true for both group algebra + univ. env. alg. is true for all co-commutative Hopf algebras.

- See Sweedler's book for attempts to classify finite-dimensional Hopf algebras (some important open questions remain)

- non-commutative, non-cocomm. Hopf algebras in short supply, before

Drinfeld-Jimbo's constructions for Lie algebras.

Simplest non-comm., non-cocomm. example: $H := U_q(\mathfrak{b}_+)$ generated by $1, X, g, g^{-1}$

with rel's: $g \cdot g^{-1} = 1 = g^{-1} \cdot g$ $gX = qXg$ ($q \in k^\times$)

with $\Delta X = X \otimes 1 + g \otimes X$ $\varepsilon(X) = 0$ $S(X) = -g^{-1}X$
 $\Delta g = g \otimes g$ $\Delta g^{-1} = g^{-1} \otimes g^{-1}$ $\varepsilon(g) = \varepsilon(g^{-1}) = 1$ $S(g) = g^{-1}$
 $S(g^{-1}) = g$
 (check $S^2 X = \bar{g}^{-1} X$)

You can think of $U_q(b^+)$ as deforming $U(b^+)$ where b^+ is Lie

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$$\text{algebra of } B^+ = \text{upper triangular matrices in } SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a, b \in \mathbb{C}, a \neq 0 \right\}$$

Good exercise: Check this is a non-comm, non-cocomm. Hopf algebra.

But at the moment, this is just an ad-hoc construction and we'd prefer some understanding of where it came from and why.