Last time, defined bialgebra - an algebra (m, η) and a coalgebra (Δ, ε) with compatible def'ns. (expressed via commutative diagrams)

Example : kG is bialgebra with $m(e_g, e_h) = e_{g*h}$, $\Delta(e_g) = e_g \otimes e_g$

check, for example, that $\underbrace{\Delta(m(e_g, e_h))}_{\substack{\Delta(e_{g*h}) \\ = e_{g*h} \otimes e_{g*h}}} = \underbrace{m(\Delta(e_g), \Delta(e_h))}_{m((e_g \otimes e_g) \otimes (e_h \otimes e_h))}$

$= e_{g*h} \otimes e_{g*h}$. ✓

m means $id \otimes \tau \otimes id$, then $m \otimes m$.

As some of us discussed after class on Friday, given any set S, form a coalgebra on vector space $k^{|S|}$ with same definitions. But in case of kG, get bialgebra.

Nevertheless, subset of elts. $x \in H$ with $\Delta(x) = x \otimes x$ will be important in sequel. "group-like elements"

One nice property of bialgebra H - Given algebra modules V, W, then $V \otimes W$ has natural $H \otimes H$ - module structure. But if we compose with coproduct $\Delta : H \to H \otimes H$, we get natural H-module

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So you may have seen G-action on $V \otimes W$ given by $g \cdot (v \otimes w) = gv \otimes gw$ (or equivalently kG action)

now understand that as coproduct in kG $\Delta : g \mapsto g \otimes g$.

Similarly tensor algebra has action from complicated Δ structure.

One further axiom to get from bialgebra to Hopf algebra –

Antipode map $S : H \to H$ with $m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon$

$$= m \circ (id \otimes S) \circ \Delta$$

or in diagrams:

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] \arrow[d, "\Delta"'] & k \arrow[r, "\eta"] & H \\ H \otimes H \arrow[rr, "id \otimes S", "S \otimes id"'] & & H \otimes H \arrow[u, "m"'] \end{tikzcd}$$

counit ε, unit η

(two diagrams)

kG example: $S : e_g \mapsto e_{g^{-1}}$ (indeed, think of antipode as a "quantum gp" version of inversion)

$T(V)$ example: $S : v \mapsto -v$ in $T^1(V)$, extend.

note – not requiring $S^2 = id$.

Proposition on antipodes S: S is an antialgebra map $S(hg) = S(g)S(h)$, $S(1_H) = 1_H$. $\forall h, g$.

and anti coalgebra map

$$(S \otimes S) \circ \Delta = \tau \circ \Delta \circ S$$

$$\varepsilon \circ S = \varepsilon.$$

Moreover S is unique.

pf: For example, for uniqueness, mimic pf. that gp. inverses are unique.

Suppose there were two: S, S'

then $S'(h) = S'(h_{(1)}\varepsilon(h_{(2)})) = S'(h_{(1)})\,\varepsilon(h_{(2)})$ (linearity)

counit axiom:
$$\begin{tikzcd} & H \otimes H \arrow[dl, "id \otimes \varepsilon"'] \\ H \otimes k \arrow[r, phantom, "\simeq"] & H \arrow[u, "\Delta"'] \end{tikzcd}$$

$$= S'(h_{(1)})\, \underbrace{m \circ (id \otimes S)(h_{(2)(1)} \otimes h_{(2)(2)})}_{m(h_{(2)(1)},\, Sh_{(2)(2)})}$$

antipode axiom: $= m \circ id \otimes S \circ \Delta$

$$\begin{aligned}
\text{Had} &= S'(h_{(1)})\, m(h_{(2)(1)}, Sh_{(2)(2)}) \\
&= S'(h_{(1)(1)})\, h_{(1)(2)}\, Sh_{(2)} && (\text{coassociativity}) \\
&= \varepsilon(h_{(1)})\, Sh_{(2)} && (\text{antipode axiom}) \\
&= S(h) && (\text{counit axiom})
\end{aligned}$$

Coassociativity in Sweedler notation: $c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$

Further adjects: commutativity / cocommutativity.

multiplication is commutative.

In diagrams.

$$\begin{tikzcd} H \otimes H \arrow[r, "\tau"] \arrow[d, "m"'] & H \otimes H \arrow[dl, "m"] \\ H & \end{tikzcd}$$

where $\tau : h_1 \otimes h_2 \mapsto h_2 \otimes h_1$.

and for cocommutative, as usual, we reverse the arrows:

$$\tau \circ \Delta = \Delta \quad \text{on} \quad H \to H \otimes H.$$

Some things are immediately nicer:

Proposition: If H is commutative or cocommutative, then $S^2 = \mathrm{id}$.

Check: kG, $T(V)$ are cocommutative.

($e_g \cdot e_h = e_{g*h}$, so kG commutative if G is abelian. check that $T(V)$ is only commutative if $\dim(V) \leq 1$.)

Immediate from defn — If H is cocommutative then H-module structure on $V \otimes W$ is isomorphic to that of $W \otimes V$.

Another example: $\mathfrak{g}$: Lie algebra — $\mathfrak{g}$ is a vector space with a

Lie bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying Jacobi intentity (J) + antisymmetry (A) props (if $\mathrm{char}(k) \neq 2$)

then $\mathcal{U}(\mathfrak{g})$ is quotient of $T(\mathfrak{g})$: tensor Hopf algebra on vector space $\mathfrak{g}$

(J): $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

(A): $[x,y] = -[y,x]$

with relation $g_1 \otimes g_2 - g_2 \otimes g_1 = [g_1, g_2]$

This gives Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$, which remains cocommutative.

(Hopf's initial motivation — unified treatment for groups, Lie algebras particularly their cohomologies)

Majid's comments:

- Any theorem true for both group algebra + univ. env. alg. is true for all co-commutative Hopf algebras.

- See Sweedler's book for attempts to classify finite-dimensional Hopf algebras (some important open questions remain)

- non-commutative, non-cocomm. Hopf algebras in short supply, before Drinfeld-Jimbo's constructions for Lie algebras.

Simplest non-comm., non-cocomm. example: $H := \mathcal{U}_q(b_+)$ generated by $1, X, g, g^{-1}$

with relns: $g \cdot g^{-1} = 1 = g^{-1} g \qquad gX = q X g \qquad (q \in k^\times)$

with $\Delta X = X \otimes 1 + g \otimes X$

$\Delta g = g \otimes g, \quad \Delta g^{-1} = g^{-1} \otimes g^{-1}$

$\varepsilon(X) = 0$

$\varepsilon(g) = \varepsilon(g^{-1}) = 1$

$S(X) = -g^{-1} X$

$S(g) = g^{-1}$

$S(g^{-1}) = g$

(check $S^2 X = q^{-1} X$)

You can think of $U_q(b^+)$ as deforming $U(b^+)$ where b^+ is Lie algebra of B^+ = upper triangular matrices in $SL_2(\mathbb{C})$ $= \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \ \begin{matrix} a, b \in \mathbb{C} \\ a \neq 0 \end{matrix} \right\}$

Good exercise: Check this is a non-comm, non-cocomm. Hopf algebra.

But at the moment, this is just an ad-hoc construction and we'd prefer some understanding of where it came from and why.