5.8 Theorem. *Let G be an open subset of the plane and $f: G \to \mathbb{C}$ an analytic function. If $\gamma_1, \ldots, \gamma_m$ are closed rectifiable curves in G such that $n(\gamma_1; w) + \cdots + n(\gamma_m; w) = 0$ for all w in $\mathbb{C} - G$ then for a in $G - \{\gamma\}$ and $k \geq 1$*

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j; a) = k! \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}}\, dz.$$

Proof. This follows immediately by differentiating both sides of the formula in Theorem 5.6 and applying Lemma 5.1. ■

5.9 Corollary. *Let G be an open set and $f: G \to \mathbb{C}$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ then for a in $G - \{\gamma\}$*

$$f^{(k)}(a) n(\gamma; a) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{k+1}}\, dz.$$

Cauchy's Theorem and Integral Formula is the basic result of complex analysis. With a result that is so fundamental to an entire theory it is usual in mathematics to seek the outer limits of the theorem's validity. Are there other functions that satisfy $\int_\gamma f = 0$ for all closed curves γ? The answer is no as the following converse to Cauchy's Theorem shows.

A closed path T is said to be *triangular* if it is polygonal and has three sides.

5.10 Morera's Theorem. *Let G be a region and let $f: G \to \mathbb{C}$ be a continuous function such that $\int_T f = 0$ for every triangular path T in G; then f is analytic in G.*

Proof. First observe that f will be shown to be analytic if it can be proved that f is analytic on each open disk contained in G. Hence, without loss of generality, we may assume G to be an open disk; suppose $G = B(a; R)$.

Use the hypothesis to prove that f has a primitive. For z in G define $F(z) = \int_{[a,z]} f$. Fix z_0 in G; then for any point z in G the hypothesis gives that $F(z) = \int_{[a,z_0]} f + \int_{[z_0,z]} f$. Hence

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{z - z_0} \int_{[z_0,z]} f.$$

This gives

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{(z - z_0)} \int_{[z_0,z]} f - f(z_0)$$

$$= \frac{1}{(z - z_0)} \int_{[z_0,z]} [f(w) - f(z_0)]\, dw.$$

$$\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \sup\{|f(w) - f(z_0)| : w \in [z, z_0]\}$$

which shows that

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = f(z_0). \blacksquare$$

Exercises

1. Suppose $f: G \to \mathbb{C}$ is analytic and define $\varphi: G \times G \to \mathbb{C}$ by $\varphi(z,w) = [f(z) - f(w)](z-w)^{-1}$ if $z \neq w$ and $\varphi(z,z) = f'(z)$. Prove that φ is continuous and for each fixed w, $z \to \varphi(z,w)$ is analytic.
2. Give the details of the proof of Theorem 5.6.
3. Let $B_\pm = \bar{B}(\pm 1; \frac{1}{2})$, $G = B(0; 3) - (B_+ \cup B_-)$. Let $\gamma_1, \gamma_2, \gamma_3$ be curves whose traces are $|z-1| = 1$, $|z+1| = 1$, and $|z| = 2$, respectively. Give γ_1, γ_2, and γ_3 orientations such that $n(\gamma_1; w) + n(\gamma_2; w) + n(\gamma_3; w) = 0$ for all w in $\mathbb{C} - G$.
4. Show that the Integral Formula follows from Cauchy's Theorem.
5. Let γ be a closed rectifiable curve in $\mathbb{C}$ and $a \notin \{\gamma\}$. Show that for $n \geq 2$ $\int_\gamma (z-a)^{-n}\, dz = 0$.
6. Let f be analytic on $D = B(0; 1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.
7. Let $\gamma(t) = 1 + e^{it}$ for $0 \leq t \leq 2\pi$. Find $\int_\gamma \left(\frac{z}{z-1} \right)^n dz$ for all positive integers n.
8. Let G be a region and suppose $f_n: G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f: G \to \mathbb{C}$ Show that f is analytic.
9. Show that if $f: \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1, 1]$ then f is an entire function.
10. Use Cauchy's Integral Formula to prove the Cayley–Hamilton Theorem: If A is an $n \times n$ matrix over $\mathbb{C}$ and $f(z) = \det(z - A)$ is the characteristic polynomial of A then $f(A) = 0$. (This exercise was taken from a paper by C. A. McCarthy, *Amer. Math. Monthly*, **82** (1975), 390–391).

§6 *The homotopic version of Cauchy's Theorem and simple connectivity*

This section presents a condition on a closed curve γ such that $\int_\gamma f = 0$ for an analytic function. This condition is less general but more geometric than the winding number condition of Theorem 5.7. This condition is also used to introduce the concept of a simply connected region; in a simply connected region Cauchy's Theorem is valid for every analytic function and every closed rectifiable curve. Let us illustrate this condition by considering a closed rectifiable curve in a disk, a region where Cauchy's Theorem is always valid (Proposition 2.15).

Let $G = B(a;\, R)$ and let $\gamma : [0,\, 1] \rightarrow G$ be a closed rectifiable curve. If $0 \le t \le 1$ and $0 \le s \le 1$, and we put $z = ta + (1-t)\gamma(s)$; then z lies on the straight line segment from a to $\gamma(s)$. Hence, z must lie in G. Let $\gamma_t(s) = ta + (1-t)\gamma(s)$ for $0 \le s \le 1$ and $0 \le t \le 1$. So, $\gamma_0 = \gamma$ and γ_1 is the curve constantly equal to a; the curves γ_t are somewhere in between. We were able to draw γ down to a because there were no holes. If a point inside γ were missing from G (imagine a stick protruding up from the disk with its base inside γ), then as γ shrinks it would get caught on the hole and could not go to the constant curve.

6.1 Definition. Let $\gamma_0, \gamma_1 : [0,\, 1] \rightarrow G$ be two closed rectifiable curves in a region G; then γ_0 is *homotopic* to γ_1 in G if there is a continuous function $\Gamma : [0,\, 1] \times [0,\, 1] \rightarrow G$ such that

6.2 $$\begin{cases} \Gamma(s,0) = \gamma_0(s) \quad \text{and} \quad \Gamma(s,1) = \gamma_1(s) \quad (0 \le s \le 1) \\ \Gamma(0,t) = \Gamma(1,t) \quad (0 \le t \le 1) \end{cases}$$

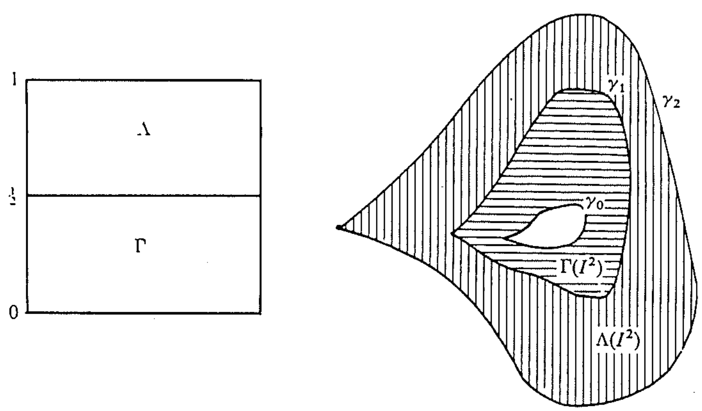
So if we define $\gamma_t : [0,\, 1] \rightarrow G$ by $\gamma_t(s) = \Gamma(s,t)$ then each γ_t is a closed curve and they form a continuous family of curves which start at γ_0 and go to γ_1. Notice however that there is no requirement that each γ_t be rectifiable. In practice when γ_0 and γ_1 are rectifiable (or smooth) each of the γ_t will also be rectifiable (or smooth).

If γ_0 is homotopic to γ_1 in G write $\gamma_0 \sim \gamma_1$. Actually a notation such as $\gamma_0 \sim \gamma_1(G)$ should be used because of the role of G. If the range of Γ is not required to be in G then, as we shall see shortly, all curves would be homotopic. However, unless there is the possibility of confusion, we will only write $\gamma_0 \sim \gamma_1$.

It is easy to show that "$\sim$" is an equivalence relation. Clearly any curve is homotopic to itself. If $\gamma_0 \sim \gamma_1$ and $\Gamma : [0,\, 1] \times [0,\, 1] \rightarrow G$ satisfies (6.2) then define $\Lambda(s,t) = \Gamma(s, 1-t)$ to see that $\gamma_1 \sim \gamma_0$. Finally, if $\gamma_0 \sim \gamma_1$ and $\gamma_1 \sim \gamma_2$ with Γ satisfying (6.2) and $\Lambda : [0,\, 1] \times [0,\, 1] \rightarrow G$ satisfying $\Lambda(s,0) = \gamma_1(s)$, $\Lambda(s,1) = \gamma_2(s)$, and $\Lambda(0,\, t) = \Lambda(1,\, t)$ for all s and t; define $\Phi : [0,1] \times [0,1] \rightarrow G$ by

$$\Phi(s,t) = \begin{cases} \Gamma(s,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \Lambda(s,2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Then Φ is continuous and shows that $\gamma_0 \sim \gamma_2$. The reader can best understand the motivation for the definition of Φ by consulting the figure above, where $I^2 = [0,1] \times [0,1]$.



6.3 Definition. A set G is *convex* if given any two points a and b in G the line segment joining a and b, $[a,b]$, lies entirely in G. The set G is *star shaped* if there is a point a in G such that for each z in G, the line segment $[a,z]$ lies entirely in G. Clearly each convex set is star shaped but the converse is just as clearly false.

We will say that G is $a-$ star shaped if $[a,z] \subset G$ whenever $z \in G$. If G is $a-$ star shaped and z and w are points in G then $[z,a,w]$ is a polygon in G connecting z and w. Hence, each star shaped set is connected.

6.4 Proposition. *Let G be an open set which is $a-$ star shaped. If γ_0 is the curve which is constantly equal to a then every closed rectifiable curve in G is homotopic to γ_0.*

Proof. Let γ_1 be a closed rectifiable curve in G and put $\Gamma(s,t) = t\gamma_1(s) + (1-t)a$. Because G is $a-$ star shaped, $\Gamma(s,t) \in G$ for $0 \le s, t \le 1$. It is easy to see that Γ satisfies (6.2). ∎

The situation in which a curve is homotopic to a constant curve is one that we will often encounter. Hence it is convenient to introduce some new terminology.

6.5 Definition. If γ is a closed rectifiable curve in G then γ is *homotopic to zero* ($\gamma \sim 0$) if γ is homotopic to a constant curve.

6.6 Cauchy's Theorem (Second Version). *If $f : G \rightarrow \mathbb{C}$ is an analytic function and γ is a closed rectifiable curve in G such that $\gamma \sim 0$, then*

$$\int_\gamma f = 0.$$

This version of Cauchy's Theorem would follow immediately from the first version if it could be shown that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ whenever $\gamma \sim 0$. This can be done. A plausible argument proceeds as follows.

Let $\gamma_1 = \gamma$ and let γ_0 be a constant curve such that $\gamma_1 \sim \gamma_0$. Let Γ satisfy (6.2) and define $h(t) = n(\gamma_t; w)$, where $\gamma_t(s) = \Gamma(s,t)$ for $0 \le s,\, t \le 1$ and w is fixed in $\mathbb{C} - G$. Now show that h is continuous on $[0,\, 1]$. Since h is integer valued and $h(0) = 0$ it must be that $h(t) \equiv 0$. In particular, $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$.

The only point of difficulty with this argument is that for $0 < t < 1$ it may be that γ_t is not rectifiable.

As was stated after Definition 6.1, in practice each of the curves γ_t will not only be rectifiable but also smooth. So there is justification in making this assumption and providing the details to transform the preceding paragraph into a legitimate proof (Exercise 9). Indeed, in a course designed for physicists and engineers this is probably preferable. But this is not desirable for the training of mathematicians.

The statement of a theorem is not as important as its proof. Proofs are important in mathematics for several reasons, not the least of which is that a proof deepens our insight into the meaning of the theorems and gives a natural delineation of the extent of the theorem's validity. Most important for the education of a mathematician, it is essential to examine other proofs because they reveal methods.

A good method is worth a thousand theorems. Not only is this statement valid as a value judgement, but also in a literal sense. An important method can be reused in other situations to obtain further results.

With this in mind a complete proof of Theorem 6.6 will be presented. In fact, we will prove a somewhat more general fact since the proof of this new result necessitates only a little more effort than the proof that $n(\gamma; w)=0$ for w in $\mathbb{C}-G$ whenever $\gamma\sim 0$. In fact, the proof of the next result more clearly reveals the usefulness of the method.

6.7 Cauchy's Theorem (Third Version). *If γ_0 and γ_1 are two closed rectifiable curves in G and $\gamma_0\sim\gamma_1$ then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for every function f analytic on G.

Before proceeding let us consider a special case. Suppose Γ satisfies (6.2) and also suppose Γ has continuous second partial derivatives. Hence

$$\frac{\partial^2\Gamma}{\partial s\partial t} = \frac{\partial^2\Gamma}{\partial t\partial s}$$

throughout the square $I^2 = [0, 1]\times[0, 1]$. Define

$$g(t) = \int_0^1 f(\Gamma(s, t))\frac{\partial\Gamma}{\partial s}(s, t)\, ds;$$

then $g(0) = \int_{\gamma_0} f$ and $g(1) = \int_{\gamma_1} f$. By Leibniz's rule g has a continuous derivative,

$$g'(t) = \int_0^1 \left[f'(\Gamma(s, t))\frac{\partial\Gamma}{\partial s}\frac{\partial\Gamma}{\partial t} + f(\Gamma(s, t))\frac{\partial^2\Gamma}{\partial t\partial s}\right] ds$$

But

$$\frac{\partial}{\partial s}\left[(f\circ\Gamma)\frac{\partial\Gamma}{\partial t}\right] = (f'\circ\Gamma)\frac{\partial\Gamma}{\partial s}\frac{\partial\Gamma}{\partial t} + f\circ\Gamma\frac{\partial^2\Gamma}{\partial s\partial t};$$

hence

$$g'(t) = f(\Gamma(1, t))\frac{\partial\Gamma}{\partial t}(1, t) - f(\Gamma(0, t))\frac{\partial\Gamma}{\partial t}(0, t).$$

Since $\Gamma(1, t) = \Gamma(0, t)$ for all t we get $g'(t) = 0$ for all t. So g is a constant; in particular $\int_{\gamma_0} f = \int_{\gamma_1} f$.

Proof of Theorem 6.7. Let $\Gamma: I^2\to G$ satisfy (6.2). Since Γ is continuous and I^2 is compact, Γ is uniformly continuous and $\Gamma(I^2)$ is a compact subset of G. Thus

$$r = d(\Gamma(I^2), \mathbb{C}-G) > 0$$

and there is an integer n such that if $(s-s')^2+(t-t')^2 < 4/n^2$ then

$$|\Gamma(s, t)-\Gamma(s', t')| < r.$$

Let

$$Z_{jk} = \Gamma\left(\frac{j}{n}, \frac{k}{n}\right), 0 \le j, k \le n$$

and put

$$J_{jk} = \left[\frac{j}{n}, \frac{j+1}{n}\right] \times \left[\frac{k}{n}, \frac{k+1}{n}\right]$$

for $0 \le j, k \le n-1$. Since the diameter of the square J_{jk} is $\frac{\sqrt{2}}{n}$, it follows that $\Gamma(J_{jk}) \subset B(Z_{jk}; r)$. So if we let P_{jk} be the closed polygon $[Z_{jk}, Z_{j+1,k}, Z_{j+1,k+1}, Z_{j,k+1}, Z_{jk}]$; then, because disks are convex, $P_{jk} \subset B(Z_{jk}; r)$. But from Proposition 2.15 it is known that

6.8
$$\int_{P_{jk}} f = 0$$

for any function f analytic in G.

It can now be shown that $\int_{\gamma_0} f = \int_{\gamma_1} f$ by going up the ladder we have constructed, one rung at a time. That is, let Q_k be the closed polygon $[Z_{0,k}, Z_{1,k} \dots, Z_{nk}]$. We will show that $\int_{\gamma_0} f = \int_{Q_0} f = \int_{Q_1} f = \cdots = \int_{Q_n} f = \int_{\gamma_1} f$ (one rung at a time!). To see that $\int_{\gamma_0} f = \int_{Q_0} f$ observe that if $\sigma_j(t) = \gamma_0(t)$ for

$$\frac{j}{n} \le t \le \frac{j+1}{n}$$

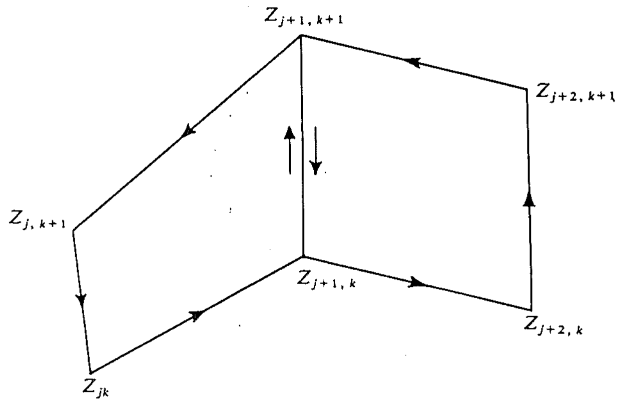
then $\sigma_j + [Z_{j+1,0}, Z_{j0}]$ (the + indicating that σ_j is to be followed by the polygon) is a closed rectifiable curve in the disk $B(Z_{j0}; r) \subset G$. Hence

$$\int_{\sigma_j} f = -\int_{[Z_{j+1,0}, Z_{j0}]} f = \int_{[Z_{j0}, Z_{j+1,0}]} f.$$

Adding both sides of this equation for $0 \le j < n$ yields $\int_{\gamma_0} f = \int_{Q_0} f$. Similarly $\int_{\gamma_1} f = \int_{Q_n} f$.

To see that $\int_{Q_k} f = \int_{Q_{k+1}} f$ use equation (6.8); this gives

6.9 $$0 = \sum_{j=0}^{n-1} \int_{P_{jk}} f$$



However, notice that the integral $\int_{P_{jk}} f$ includes the integral over $[Z_{j+1,k}, Z_{j+1,k+1}]$, which is the negative of the integral over $[Z_{j+1,k+1}, Z_{j+1,k}]$, which is part of the integral $\int_{P_{j+1,k}} f$. Also,

$$Z_{0,k} = \Gamma\left(0, \frac{k}{n}\right) = \Gamma\left(1, \frac{k}{n}\right) = Z_{n,k}$$

so that $[Z_{0,k+1}, Z_{0,k}] = -[Z_{1k}, Z_{1,k+1}]$. Hence, taking these cancellations into consideration, equation (6.9) becomes

$$0 = \int_{Q_k} f - \int_{Q_{k+1}} f$$

This completes the proof of the theorem. ■

The second version of Cauchy's Theorem immediately follows by letting γ_1 be a constant path in (6.7).

6.10 Corollary. *If γ is a closed rectifiable curve in G such that $\gamma \sim 0$ then $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$.*

The converse of the above corollary is not valid. That is, there is a closed rectifiable curve γ in a region G such that $n(\gamma; w) = 0$ for all w in $\mathbb{C} - G$ but γ is not homotopic to a constant curve (Exercise 8).

If G is an open set and γ_0 and γ_1 are closed rectifiable curves in G then $n(\gamma_0; a) = n(\gamma_1; a)$ for each a in $\mathbb{C} - G$ provided $\gamma_0 \sim \gamma_1(G)$. Let $\gamma_0(t) = e^{2\pi it}$ and $\gamma_1(t) = e^{-2\pi it}$ for $0 \le t \le 1$. Then $n(\gamma_0; 0) = 1$ and $n(\gamma_1; 0) = -1$ so that γ_0 and γ_1 are not homotopic in $\mathbb{C} - \{0\}$.

6.11 Definition. If $\gamma_0, \gamma_1 : [0, 1] \to G$ are two rectifiable curves in G such that $\gamma_0(0) = \gamma_1(0) = a$ and $\gamma_0(1) = \gamma_1(1) = b$ then γ_0 and γ_1 are *fixed-end-point*

homotopic (FEP homotopic) if there is a continuous map $\Gamma: I^2 \to G$ such that

6.12 $$\Gamma(s, 0) = \gamma_0(s) \qquad \Gamma(s, 1) = \gamma_1(s)$$
$$\Gamma(0, t) = a \qquad \Gamma(1, t) = b$$

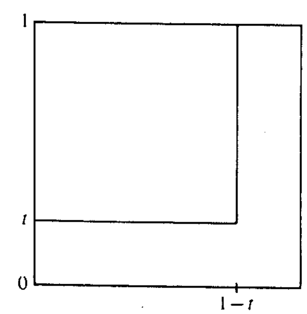
for $0 \le s, t \le 1$.

Again the relation of FEP homotopic is an equivalence relationship on curves from one given point to another (Exercise 3).

Notice that if γ_0 and γ_1 are rectifiable curves from a to b then $\gamma_0 - \gamma_1$ is a closed rectifiable curve. Suppose Γ satisfies (6.12) and define $\gamma: [0, 1] \to G$ by $\gamma(s) = \gamma_0(3s)$ for $0 \le s \le \frac{1}{3}$; $\gamma(s) = b$ for $\frac{1}{3} \le s \le \frac{2}{3}$; and $\gamma(s) = \gamma_1(3-3s)$ for $\frac{2}{3} \le s \le 1$. We will show that $\gamma \sim 0$. In fact, define $\Lambda: I^2 \to G$ by

$$\Lambda(s, t) = \begin{cases} \Gamma(3s(1-t), t) & \text{if } 0 \le s \le \frac{1}{3} \\ \Gamma(1-t, 3s-1+2t-3st) & \text{if } \frac{1}{3} \le s \le \frac{2}{3} \\ \gamma_1((3-3s)(1-t)) & \text{if } \frac{2}{3} \le s \le 1. \end{cases}$$

Although this formula may appear mysterious it can easily be understood by seeing that for a given value of t, Λ is the restriction of Γ to the boundary of the square $[0, 1-t] \times [t, 1]$ (see the figure). It is left to the reader to check that Λ shows $\gamma \sim 0$.



Hence, for f analytic on G the second version of Cauchy's Theorem gives

$$0 = \int_\gamma f = \int_{\gamma_0} f - \int_{\gamma_1} f.$$

This is summarized in the following.

6.13 Independence of Path Theorem. *If γ_0 and γ_1 are two rectifiable curves in G from a to b and γ_0 and γ_1 are FEP homotopic then*

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

for any function f analytic in G.

Those regions G for which the integral of an analytic function around a closed curve is always zero can be characterized.

6.14 Definition. An open set G is *simply connected* if G is connected and every closed curve in G is homotopic to zero.

6.15 Cauchy's Theorem (Fourth Version). *If G is simply connected then $\int_\gamma f=0$ for every closed rectifiable curve and every analytic function f.*

Let us now take a few moments to digest the concept of simple connectedness. Clearly every star shaped region is simply connected. Also, examine the complement of the spiral $r = \theta$. That is, let $G = \mathbb{C}-\{\theta e^{i\theta}: 0 \le \theta < \infty\}$; then G is simply connected. In fact, it is easily seen that G is open and connected. If one argues in an intuitive way it is not difficult to become convinced that every curve in G is homotopic to zero. A rigorous proof will be postponed until we have proved the following: *A region G is simply connected iff $\mathbb{C}_\infty - G$, its complement in the extended plane, is connected in $\mathbb{C}_\infty$.* This will not be proved until Chapter VIII. If this criterion is applied to the region G above then G is simply connected since $\mathbb{C}_\infty - G$ consists of the spiral $r = \theta$ and the point at infinity.

Notice that for $G = \mathbb{C}-\{0\}$, $\mathbb{C}-G = \{0\}$ is connected but $\mathbb{C}_\infty - G = \{0, \infty\}$ is not. Also, the domain of the principal branch of the logarithm is simply connected.

It was shown earlier in this chapter (Corollary 1.22) that if an analytic function f has a primitive in a region G then the integral of f around every closed rectifiable curve in G is zero. The next result should not be too surprising in light of this.

6.16 Corollary. *If G is simply connected and $f: G \to \mathbb{C}$ is analytic in G then f has a primitive in G.*

Proof. Fix a point a in G and let γ_1, γ_2 be any two rectifiable curves in G from a to a point z in G. (Since G is open and connected there is always a path from a to any other point of G.) Then, by Theorem 6.15, $0 = \int_{\gamma_1-\gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f$ (where $\gamma_1 - \gamma_2$ is the curve which goes from a to z along γ_1 and then back to a along $-\gamma_2$). Hence we can get a well defined function F: $G \to \mathbb{C}$ by setting $F(z) = \int_\gamma f$ where γ is any rectifiable curve from a to z. We claim that F is a primitive of f.

If $z_0 \in G$ and $r > 0$ is such that $B(z_0; r) \subset G$, then let γ be a path from a to z_0. For z in $B(z_0; r)$ let $\gamma_z = \gamma + [z_0, z]$; that is, γ_z is the path γ followed by the straight line segment from z_0 to z. Hence

$$\frac{F(z)-F(z_0)}{z-z_0} = \frac{1}{(z-z_0)} \int_{[z_0,z]} f.$$

Now proceed as in the proof of Morera's Theorem to show that $F'(z_0)=f(z_0)$. ■

Perhaps a somewhat less expected consequence of simple connectedness is the fact that a branch of $\log f(z)$ where f is analytic and never vanishes, can be defined on a simply connected region. Nevertheless this is a direct consequence of the preceding corollary.

6.17 Corollary. *Let G be simply connected and let $f: G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any z in G. Then there is an analytic function*

$g: G \to \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Proof. Since f never vanishes, $\dfrac{f'}{f}$ is analytic on G; so, by the preceding corollary, it must have a primitive g_1. If $h(z) = \exp g_1(z)$ then h is analytic and never vanishes. So, $\dfrac{f}{h}$ is analytic and its derivative is

$$\frac{h(z)f'(z)-h'(z)f(z)}{h(z)^2}$$

But $h' = g_1'h$ so that $hf' - fh' = 0$. Hence f/h is a constant c for all z in G. That is $f(z) = c \exp g_1(z) = \exp [g_1(z)+c']$ for some c'. By letting $g(z) = g_1(z)+c'+2\pi ik$ for an appropriate k, $g(z_0) = w_0$ and the theorem is proved. ■

Let us emphasize that the hypothesis of simple connectedness is a topological one and this was used to obtain some basic results of analysis. Not only are these last three theorems (6.15, 6.16, and 6.17) consequences of simple connectivity, but they are equivalent to it. It will be shown in Chapter VIII that if a region G has the conclusion of each of these theorems satisfied for every function analytic on G, then G must be simply connected.

We close this section with a definition.

6.18 Definition. If G is an open set then γ is *homologous to zero*, in symbols $\gamma \approx 0$, if $n(\gamma; w) = 0$ for all w in $\mathbb{C}-G$.

Using this notation, Corollary 6.10 says that $\gamma \sim 0$ implies $\gamma \approx 0$. This result appears in Algebraic Topology when it is shown that the first homology group of a space is isomorphic to the abelianization of the fundamental group. In fact, those familiar with homology theory will recognize in the proof of Theorem 6.7 the elements of simplicial approximation.

Exercises

1. Let G be a region and let σ_1, σ_2: $[0,1] \to G$ be the constant curves $\sigma_1(t) \equiv a$, $\sigma_2(t) \equiv b$. Show that if γ is a closed rectifiable curve in G and $\gamma \sim \sigma_1$ then $\gamma \sim \sigma_2$. (Hint: connect a and b by a curve.)
2. Show that if we remove the requirement "$\Gamma(0, t) = \Gamma(1, t)$ for all t" from Definition 6.1 then the curve $\gamma_0(t) = e^{2\pi it}$, $0 \le t \le 1$, is homotopic to the constant curve $\gamma_1(t) \equiv 1$ in the region $G = \mathbb{C}-\{0\}$.
3. Let $\mathscr{C}$ = all rectifiable curves in G joining a to b and show that Definition 6.11 gives an equivalence relation on $\mathscr{C}$.
4. Let $G = \mathbb{C}-\{0\}$ and show that every closed curve in G is homotopic to a closed curve whose trace is contained in $\{z: |z| = 1\}$.