sec 2.1.2 # (2,4,7) pg 28 30/30

#2| verify Cauch-Riemann's equations for the functions z^2 and z^3.

+5

Pf: let $z = x+iy$ thus

$$z^2 = (x+iy)^2 = x^2 + 2ixy - y^2 = (x^2-y^2) + i2xy$$

$$z^3 = (x+iy)^3 = x^3 - 3xy^2 + 3ix^2y - iy^3$$

$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

Take $f(z) = z^2$. let $u = x^2 - y^2$ and $v = 2xy$

thus $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$

Take $f(z) = z^3$. let $u = x^3 - 3xy^2$ and $v = 3x^2y - y^3$

Thus $\frac{\partial u}{\partial x} = 3x^2 - 3y^2 = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -6xy = -\frac{\partial v}{\partial x}$ //

#41 Show that an analytic function cannot have constant absolute value without reducing to a constant.

Pf: suppose $f(z) = u + iv$. where $f: \mathbb{C} \to \mathbb{C}$

then we know then $|f(z)| = \text{constant} = C$.

thus we have $u^2 + v^2 = C^2$.

Then $\begin{cases} 2u\dfrac{\partial u}{\partial x} + 2v\dfrac{\partial v}{\partial x} = 0 \\ \\ 2u\dfrac{\partial u}{\partial y} + 2v\dfrac{\partial v}{\partial y} = 0 \end{cases}$

and

Now using the Cauch Riemann equations,

we get $\begin{cases} u\dfrac{\partial u}{\partial x} - v\dfrac{\partial u}{\partial y} = 0 \\ \\ u\dfrac{\partial u}{\partial y} + v\dfrac{\partial u}{\partial x} = 0 \end{cases}$

and

we can eliminate $\dfrac{\partial u}{\partial y}$, and so we get

$(u^2+v^2)\dfrac{\partial u}{\partial x} = 0$ $\Rightarrow$ $\dfrac{\partial u}{\partial x} = 0$ if $w = u+iv \neq 0$.

cont'd. →

"ilarly $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

since the 4 partial derivatives of u, v are zero, the functions u, v are constant and so this implies that $w = u + iv$ is a constant as well. //

#7 show that the harmonic function satisfies the formal differential equation. $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0.$

Pf: let u be a harmonic function.

Therefore we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Taking the definitions of $\frac{\partial f}{\partial z}$ and $\frac{\partial f}{\partial \bar{z}}$ from

we know $\frac{\partial u}{\partial z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)$

and $\frac{\partial u}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)$

and so we have the following:

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar{z}}\right)$$

cntd. →

$$= \frac{\partial}{\partial z}\left(\frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right)\Big)$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left(\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right] - i\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y}\right]\right)$$

$$= \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 u}{\partial x\partial y} - i\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

this is because $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

and clearly $\dfrac{i\partial^2 u}{\partial x\partial y} = \dfrac{i\partial^2 u}{\partial y\partial x}$ //.

2.1.4 #(2,3,6) pg 32

#2 If Q is a polynomial with distinct roots $\alpha_1 \ldots \alpha_n$ and if P is a polynomial of degree $<n$ show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)},$$

5/5

Pf: Each root α_i is a simple zero since there are n distinct roots. And so $Q(\alpha_i)=0$ and $Q'(\alpha_i) \neq 0$.

Thus we can rewrite $\frac{P(z)}{Q(z)}$ as a partial decomposition of fractions by

$$\frac{P(z)}{Q(z)} = \frac{P(z)}{(z-\alpha_1)\cdots(z-\alpha_n)} = \frac{A_1}{(z-\alpha_1)} + \cdots + \frac{A_n}{(z-\alpha_n)}.$$

for some $A_1, \ldots A_n$.

Now we can see that;

$$Q'(z) = [(z-\alpha_1)\cdots(z-\alpha_{i-1})(z-\alpha_i)(z-\alpha_{i+1})\cdots(z-\alpha_n)]'$$

$$= [(z-\alpha_i)(z-\alpha_1)\cdots(z-\alpha_{i-1})(z-\alpha_{i+1})\cdots(z-\alpha_n)]'$$

$$= (z-\alpha_i)'[(z-\alpha_1)\cdots(z-\alpha_{i-1})(z-\alpha_{i+1})\cdots(z-\alpha_n)]$$

$$+ (z-\alpha_i)[(z-\alpha_1)\cdots(z-\alpha_{i-1})(z-\alpha_{i+1})\cdots(z-\alpha_n)]'$$

(By the product rule for derivatives.) →

which this implies that

$$Q'(\alpha_i) = (\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)$$

Thus now we have ✓

$$\frac{P(z)}{Q(z)} = \frac{A_1}{(z-\alpha_1)} + \cdots + \frac{A_n}{(z-\alpha_n)}$$

$$\Rightarrow P(z) = \frac{A_1 Q(z)}{(z-\alpha_1)} + \cdots + \frac{A_1 Q(z)}{(z-\alpha_n)}$$

$$= A_1(z-\alpha_2)(z-\alpha_3)\cdots(z-\alpha_n)$$
$$+ A_2(z-\alpha_1)(z-\alpha_3)\cdots(z-\alpha_n)$$
$$\vdots$$
$$+ A_n(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_{n-1})$$

Now we need to solve for A_i. To do this let us evaluate $P(\alpha_i)$.

$$P(\alpha_i) = A_i(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_i-\alpha_n)$$

$$= A_i Q'(\alpha_i)$$

$$\Rightarrow A_i = \frac{P(\alpha_i)}{Q'(\alpha_i)}$$ which holds for all i where $1 \leq i \leq n$.

Thus $$\frac{P(z)}{Q(z)} = \frac{P(\alpha_1)}{Q'(\alpha_1)(z-\alpha_1)} + \frac{P(\alpha_2)}{Q'(\alpha_2)(z-\alpha_2)} + \cdots + \frac{P(\alpha_n)}{Q'(\alpha_n)(z-\alpha_n)}$$

$$= \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$$ ✓ //

Prove there exists a unique polynomial P of degree P of degree $<n$ with given values c_k at the points α_k.

Pf: we have to show existence and uniqueness.

Take $$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)(z-\alpha_k)}$$

and suppose we are given that $P(\alpha_k) = c_k$

thus $$\frac{P(z)}{Q(z)} = \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z-\alpha_k)}$$

$$\Rightarrow \quad P(z) = Q(z) \sum_{k=1}^{n} \frac{c_k}{Q'(\alpha_k)(z-\alpha_k)}$$ and $\deg(P(z)) < n$.

Thus $\exists$ such a polynomal

For uniqueness, suppose there is another polynomial $L(z)$ that also satisfies $L(z) = c_k$ and $\deg(L(z)) < n$. Then we know that $P(z) - L(z)$ is a polynomial as well and $\deg(P(z) - L(z)) < n$ and has n roots $\alpha_1 \cdots \alpha_n$.

Thus $P(z) - L(z) = 0 \Rightarrow P(z) = L(z)$.

Therefore unique.

con't'd →

suppose we look at $P(\alpha_1)$

then $P(\alpha_1) = \dfrac{c_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n)}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\cdots(\alpha_1-\alpha_n)} = c_1$

and $P(\alpha_k) = c_k \quad \forall k = 1, \ldots n$.

and so $P(z) = \displaystyle\sum_{k=1}^{n} c_k \prod_{\substack{m=1 \\ m \neq k}}^{n} \frac{z-\alpha_m}{\alpha_k-\alpha_m}$. //

If $R(z)$ is a rational function of order n, how large or small can the order of $R'(z)$ be?

Pf: Let $R(z) = \frac{P(z)}{Q(z)}$, where $R(z)$ is of order n.

Then if $n=0$, $R(z)$ has no zeros, and $R'(z)=0$. Thus the order of $R'(z)$ is undefined.

Now suppose $n \geq 1$. Thus we have two cases.

case 1: suppose that $R(z)$ has no pole. Therefore $R(z)$ is a normal real polynomial of degree n and thus $R'(z)$ is of degree $n-1$ so the order is $n-1$.

case 2: if $R(z)$ has at least one complex pole, then we have that $n \leq \text{ord}(R'(z)) \leq 2n$.

we can see this by writing

$$R(z) = \frac{P(z)}{Q(z)} = \frac{P(z)}{(z-a_1)^{n_1} \cdots (z-a_m)^{n_m}}$$ By the fundamental Theorem of Algebra

and $P(z)$ and $Q(z)$ have no common factors.

and $n_i, m \geq 1$ and a_i distinct

Thus $Q'(z) = (z-\alpha_1)^{n_1-1} \cdots (z-\alpha_m)^{n_m-1} \overbrace{[n_1(z-\alpha_2)\cdots(z-\alpha_m) + \cdots + n_m(z-\alpha_1)\cdots(z-\alpha_{m-1})]}^{(*)} \to$

By the product rule.

Thus by the Quotient rule we have

$$R'(z) = \frac{(z-a_1)^{n_1-1}\cdots(z-a_m)^{n_m-1}\left[\left(P'(z)\left((z-a_1)\cdots(z-a_m)\right)\right) - P(z)(*)\right]}{(z-a_1)^{2n_1}\cdots(z-a_m)^{2n_m}}$$

$$= \frac{(z-a_1)\cdots(z-a_m)P'(z) - P(z)(*)}{(z-\alpha_1)^{n_1+1}\cdots(z-\alpha_m)^{n_m+1}}$$

Then call t degree of $P(z)$ and s degree of $Q(z)$

thus the order of $R(z)$ is $\max\{s,t\}$.

the order of the numerator in $R'(z)$ is $m+s-1$

and the order of the denominator is $t+m$.

Thus the order of $R'(z)$ is

$$\begin{aligned}
\text{order} &= \max\{\text{deg numerator}, \text{deg denominator}\}\\
&\leq \max\{m+s-1, t+m\}\\
&= \max\{s-1, t\}+m\\
&\leq \max\{s,t\}+m \leq 2n.
\end{aligned}$$

— this is how large the order of $R'(z)$ can be

For how small the order of $R'(z)$ can be

if $t \neq s$ then order $= \max\{s+m-1, t+m\}$

$= \max\{s, t+1\}+m-1 \geq \max\{s,t\}+m-1 \geq n$. //.

2.2.3 #(3,5) pg. 37

#3| Show that the sum of an absolutely convergent series does not change if the terms are rearranged.

Pf: Let us assume that $\sum_{n=1}^{\infty} a_n = S$ and $\sum a_n$ is absolutely convergent.

Then Given an $\varepsilon > 0$ we choose $N_1(\varepsilon)$ so that $\left|\sum_{k=1}^{n} a_k - S\right| < \varepsilon/2$ for $n \geq N_1(\varepsilon)$

and choose $N_2(\varepsilon)$ so that $\sum_{k=m}^{n} |a_k| < \varepsilon/2$ for $m,n \geq N_2(\varepsilon)$,

let us assume further that $N_2 > N_1$.

Define φ as an onto and one to one function & $\varphi: \mathbb{N} \to \mathbb{N}$ be our rearrangment.

Now choose $N(\varepsilon)$ so that

$$\{n \in \mathbb{N} : n \leq N_1(\varepsilon) \leq N_2(\varepsilon)\} \subset \{\varphi(k) : k \leq N(\varepsilon)\}$$

Note that $N(\varepsilon) \geq N_2(\varepsilon)$. Thus for $n \geq N(\varepsilon)$ we have $\left|\sum_{k=1}^{n} a_{\varphi(k)} - S\right| \leq \left|\sum_{k=1}^{N_2(\varepsilon)} a_k - S\right| + \underbrace{\sum |a_{\varphi(k)}|}_{\{\varphi(k):\ \varphi(k) > N_2(\varepsilon),\ k \leq n\}} < \varepsilon/2 + \varepsilon/2$

Thus $\lim_{n\to\infty} \sum_{k=1}^{n} a_{\varphi(k)} = S$. //

#5) Discuss the uniform convergence of the seri

$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ $\forall x \in \mathbb{R}$.

Pf: we know that $f_n(x) = \frac{x}{n(1+nx^2)}$ is an odd function with a maximum and minimum at $\pm\sqrt{1/n}$, (we know $f_n' = 0$) and $f_n(x) \to 0$ as $x \to \pm\infty$. Thus for $\forall x \in \mathbb{R}$ we have that $|f_n(x)| \leq f(\sqrt{1/n}) = \frac{1}{2} n^{-3/2}$. Since we know that $\sum \frac{1}{2} n^{-3/2}$ converges and so by the Weierstrass M-test, $\sum \frac{x}{n(1+nx^2)}$ converges uniformly. //

° 2.2.4 #(2,4,6,8) pg 41.

Expand $\frac{2z+3}{z+1}$ in powers of $z-1$. What is the radius of convergence?

Solution: $\frac{2z+3}{z+1} = \frac{2z+2+1}{z+1}$

$$= \frac{2(z+1)+1}{z+1}$$

$$= 2 + \frac{1}{z+1}$$

$$= 2 + \frac{1}{(z-1)+2}$$

$$= 2 + \frac{1}{2}\left(\frac{1}{1-\left(\frac{1-z}{2}\right)}\right)$$

$$= 2 + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1-z}{2}\right)^n = 2 + \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n (z-1)^n$$

$$= \boxed{\frac{3}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)^n (z-1)^n}$$

Clearly a power series. $a_n = \frac{1}{2}\left(-\frac{1}{2}\right)^n$

so $R = \frac{1}{\limsup \sqrt[n]{|a_n|}} = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{\frac{1}{2^{n+1}}}} = \frac{1}{\limsup_{n\to\infty} \frac{1}{2}\frac{1}{\sqrt[n]{2}}} = \frac{1}{\frac{1}{2}} = 2.$

↑ radius of convergence. //.

#4 if $\sum a_n z^n$ has radius of convergence R, what is the radius of Conv. of $\sum a_n z^{2n}$? If $\sum a$

Pf: By definition, the radius of convergence of

$\sum a_n z^n$ is $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$

$$\Rightarrow \frac{1}{R} = \limsup \sqrt[n]{|a_n|} =$$

so let R^* be the ROC of $\sum a_n z^{2n}$ then

$$\frac{1}{R^*} = \limsup \sqrt[2n]{|a_n|}$$

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$$= \limsup \left(\sqrt[n]{|a_n|}\right)^{1/2}$$

$$= \left(\frac{1}{R}\right)^{1/2} = \frac{1}{\sqrt{R}} \Rightarrow \boxed{R^* = \sqrt{R}}$$

let R^{**} be the ROC of $\sum a_n^2 z^n$ then

$$\frac{1}{R^{**}} = \limsup \sqrt[n]{|a_n|^2}$$

$$= \limsup \left(\sqrt[n]{|a_n|}\right)^2$$

$$= \left(\frac{1}{R}\right)^2 = \frac{1}{R^2} \Rightarrow \boxed{R^{**} = R^2}$$

//

if $\sum a_n z^n$ and $\sum b_n z^n$ have ROC's R_1 and R_2 resp. show the ROC of $\sum a_n b_n z^n$ is atleast $R_1 R_2$.

Pf: Suppose we take any real number c such that $0 < c < R_1 R_2$. We will show that $\sum^{\infty} a_n b_n z^n$ is convergent. Since $c < R_1 R_2$ we have that $\frac{c}{R_1} < R_2$ and $\frac{c}{R_2} < R_1$. Thus let

$$z_1 = \frac{1}{2}\left(R_1 + \frac{c}{R_2}\right) < R_1 \quad \text{and} \quad z_2 = \frac{1}{2}\left(R_2 + \frac{c}{R_1}\right) < R_2$$

Therefore we see that $\sum a_n z_1^n$ and $\sum b_n z_2^n$ are absolutely convergent, and thus the term by term product $\sum a_n z_1^n b_n z_2^n = \sum a_n b_n z_1^n z_2^n$ is absolutely convergent. We can see further that

$$z_1 z_2 = \frac{1}{4}\left(R_1 R_2 + 2c + \frac{c^2}{R_1 R_2}\right)$$

$$= \frac{c}{2} + \frac{1}{4}\left(R_1 R_2 + \frac{c^2}{R_1 R_2}\right)$$

$$> \frac{c}{2} + \frac{1}{2}\left(\sqrt{R_1 R_2} \cdot \sqrt{\frac{c^2}{R_1 R_2}}\right) = c$$

Therefore $\sum a_n b_n z^n$ is absolutely convergent. and so it is absolutely convergent for all x such that $|x| \le c$. Since c is any real # s.t. $0 < c < R_1 R_2$ this means that $\sum a_n b_n z^n$ is absolutely →

convergent for all x such that $|x| < R_1 R_2$. The the radius of convergence of $\sum a_n b_n z^n$ is at le $R_1 R_2$.

In other words

if $R_1 = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$ and $R_2 = \dfrac{1}{\limsup \sqrt[n]{|b_n|}}$

then the ROC, R, of $\sum a_n b_n z^n$ is

$$R = \frac{1}{\limsup \sqrt[n]{|a_n b_n|}}$$

$$\geq \frac{1}{\limsup \sqrt[n]{|a_n||b_n|}}$$

this is because over $\mathbb{C}$

$|xy| \geq |x||y|$

ex: take $x = i$ and $y = i$

$|i \cdot i| = |i^2| = |-1| = 1 \geq$

$|i||i| = \sqrt{-1} \cdot \sqrt{-1} = -1$

$$= \frac{1}{\limsup \sqrt[n]{|a_n|} \sqrt[n]{|b_n|}}$$

$$= \frac{1}{\limsup \sqrt[n]{|a_n|} \limsup \sqrt[n]{|b_n|}}$$

$= R_1 R_2$. //

1 For what values of z is $\sum_0^\infty \left(\frac{z}{1+z}\right)^n$ convergent?

Pf: let us rewrite as $w = \frac{z}{1+z}$

thus $\sum_0^\infty w^n$ is looks like a geometric series.

Thus this only happens when $|w| < 1$

or in other words when $\left|\frac{z}{1+z}\right| < 1$

$$\Rightarrow \frac{|z|}{|1+z|} < 1$$

$$\Rightarrow |z| < |1+z|.$$

So the series is absolutely convergent when:

$$\Rightarrow |z|^2 < |1+z|^2$$

$$\Rightarrow z\bar{z} < (1+z)(\overline{1+z})$$

$$\Rightarrow z\bar{z} < 1 + z + \bar{z} + z\bar{z}$$

$$\Rightarrow 0 < 1 + z + \bar{z}$$

$$\Rightarrow 0 < 1 + (x+iy) + (x-iy)$$

$$\Rightarrow 0 < 1 + 2\,\mathrm{Re}(z)$$

$$\Rightarrow -\frac{1}{2} < \mathrm{Re}(z) \;//$$

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