Problem 1 *(Ahlfors 4.1.3.1) Compute*

$$\int_\gamma x dz$$

where γ is the directed line segment from 0 to $1+i$.

We can represent the arc γ by

$$z(t) = (1+i)t \implies z(0) = 0, z(1) = (1+i).$$

Observe $Re(z(t)) = t$ and $z'(t) = (1+i)$. Thus

$$\begin{aligned}\int_\gamma x dz &= \int_0^1 t(1+i)dt \\ &= \frac{1+i}{2}t^2\Big|_0^1 \\ &= \frac{1+i}{2}\end{aligned}$$

Problem 2 *(Ahlfors 4.1.3.2) Compute $\int_{|z|=r} x dz$ for the positive sense of the circle, in two ways: first, by use of a parameter and second, by observing that $x = \frac{1}{2}(z+\bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$ on the circle.*

Using the first method, we represent the arc γ by $z(t) = re^{it}$, where $t \in [0, 2\pi]$. Observe $Re(z(t)) = Re(re^{it}) = r\cos(t)$ and $z'(t) = ire^{it}$. Hence

$$\begin{aligned}\int_{|z|=r} x dx &= \int_0^{2\pi} r\cos(t) ir(\cos(t) + i\sin(t))dt \\ &= ir^2 \int_0^{2\pi} (\cos(t))^2 + i\sin(t)dt \\ &= ir^2 \int_0^{2\pi} \frac{1+\cos(2t)}{2} + i\sin(t)dt \\ &= ir^2 \left(\frac{t}{2} + \frac{\sin(2t)}{4} - i\cos(t)\right)\Big|_0^{2\pi} \\ &= ir^2\left(\frac{2\pi}{2} - i + i\right) \\ &= i\pi r^2\end{aligned}$$

For the second method, using $x = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$, we have

$$\begin{aligned}
\int_{|z|=r} x\,dz &= \frac{1}{2}\int_{|z|=r}\left(z + \frac{r^2}{z}\right)dz \\
&= \frac{1}{2}\int_{|z|=r} z\,dz + \frac{r^2}{2}\int_{|z|=r}\frac{1}{z}dz \\
&= \frac{1}{2}\int_{|z|=r}\frac{1}{z}dz
\end{aligned}$$

where the last inequality comes from the discussion on page 107, $\int_\gamma z^n dz = 0$ for all closed curves γ and $n \geq 0$. To evaluate the remaining integral we again reference the discussion. For $n = -1$, over the circle, $\int_C z^n dz = 2\pi i$. Thus

$$\int_{|z|=r} x\,dz = \frac{r^2}{2}\int_{|z|=r}\frac{1}{z}dz = \frac{r^2}{2} * 2\pi i = i\pi r^2.$$

Problem 3 *(Ahlfors 4.1.3.3) Compute*

$$\int_{|z|=2}\frac{dz}{z^2-1}$$

for the positive sense of the circle.

Observe that $\frac{1}{z^2-1} = \frac{1}{(z+1)(z-1)}$. Thus by partial fraction decomposition, $\frac{1}{z^2-1} = -\frac{1}{2(z+1)} + \frac{1}{2(z-1)}$. Hence

$$\begin{aligned}
\int_{|z|=2}\frac{dz}{z^2-1} &= \int_{|z|=2} -\frac{1}{2(z+1)}dz + \int_{|z|=2}\frac{1}{2(z-1)}dz \\
&= -\frac{1}{2}\int_{|z+1|=1}\frac{dz}{z+1} + \frac{1}{2}\int_{|z-1|=1}\frac{dx}{z-1} \\
&= -\frac{2\pi i}{2} + \frac{2\pi i}{2} \\
&= 0
\end{aligned}$$

Problem 4 *(Ahlfors 4.1.3.4) Compute*

$$\int_{|z|=1}|z-1|\cdot|dz|.$$

We represent the arc γ by $z(t) = e^{it}$, for $t \in [0, 2\pi]$.

$$\begin{aligned}\int_{|z|=1} |z-1| \cdot |dz| &= \int_0^{2\pi} |e^{it} - 1| dt \\ &= \int_0^{2\pi} \sqrt{2 - 2\cos(t)} dt \\ &= \int_0^{2\pi} 2|\sin(t/2)| dt \\ &= -4\cos(t/2)\Big|_0^{2\pi} \\ &= 8\end{aligned}$$

Problem 5 *(Ahlfors 4.1.3.6) Assume that $f(z)$ is analytic and satisfies the inequality $|f(z) - 1| < 1$ in a region Ω. Show that*

$$\int_\gamma \frac{f'(z)}{f(z)} dz = 0$$

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for every closed curve in Ω. (The continuity of $f'(z)$ is taken for granted.)

Observe $\log(z)$ is single valued and analytic, $-\pi < \arg(z) < \pi$. We also have $-\pi < \arg f(z) < \pi$, and so, $\log f(z)$ is also analytic and single valued.

$$\frac{d}{dz}(\log f(z)) = \frac{1}{f(z)} f'(z) dz.$$

Since γ is a closed curve, it follows that $\int_\gamma \frac{f'(z)}{f(z)} dz = 0$.

Problem 6 *(Ahlfors 4.2.2.1) Compute*

$$\int_{|z|=1} \frac{e^z}{z} dz.$$

We recall Theorem 5 from Ahlfors which says for $f(z)$ analytic in an open disc with a finite number of points ζ_j removed and $f(z)$ satisfying $\lim_{z \to \zeta_j}(z - \zeta_j) f(z) = 0$ for all j, then $\int_\gamma f(z) dx = 0$.

Let us consider the function $f(z) = \frac{e^z - 1}{z}$, which is analytic on $\Delta' := B(0,2) \setminus \{z = 0\}$. Checking the limit condition from Theorem 5 we find our function satisfies:

$$\lim_{z \to 0}(z - 0) f(z) = \lim_{z \to 0} z \frac{e^z - 1}{z} = \lim_{z \to 0} e^z - 1 = 0.$$

Therefore, by the theorem, $\int_{|z|=1} \frac{e^z}{z} - \frac{1}{z} dz = 0$. Thus

$$\int_{|z|=1} \frac{e^z}{z} dz = \int_{|z|=1} \frac{1}{z} dz = 2\pi i.$$

Problem 7 *(Ahlfors 4.2.2.3) Compute*

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2}$$

under the condition $|a| \neq \rho$. (Hint: make use of the equations $z\bar{z} = \rho^2$ and $|dz| = -i\rho\frac{dz}{z}$.

Let us set $z(t) = \rho e^{it}$, for $t \in [0, 2\pi]$. Then $|dz| = -i\rho\frac{dz}{z} = -i\rho\frac{i\rho e^{it}dt}{\rho e^{it}} = \rho dt$. So

$$\begin{aligned}
\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} &= \int_{|z|=\rho} \frac{-i\rho dz}{(z-a)(\bar{z}-\bar{a})z} \text{ ?} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\frac{\rho^2}{z}-\bar{a})z} \\
&= -i\rho \int_{|z|=\rho} \frac{dz}{(z-a)(\rho^2 - z\bar{a})} \\
&= \frac{i\rho}{\bar{a}} \int_{|z|=\rho} \frac{dz}{(z-a)(z-\frac{\rho^2}{\bar{a}})} \\
&= \frac{i\rho}{\bar{a}(a-\frac{\rho^2}{\bar{a}})} \int_{|z|=\rho} \frac{z-\frac{\rho^2}{\bar{a}} - z + a}{(z-a)(z-\frac{\rho^2}{\bar{a}})} dz \\
&= \frac{i\rho}{|a|^2-\rho^2}\left[\int_{|z|=\rho} \frac{z-\frac{\rho^2}{\bar{a}}}{(z-a)(z-\frac{\rho^2}{\bar{a}})} dz - \int_{|z|=\rho} \frac{z-a}{(z-a)(z-\frac{\rho^2}{\bar{a}})} dz\right] \\
&= \frac{i\rho}{|a|^2-\rho^2}\left[\int_{|z|=\rho} \frac{dz}{z-a} - \int_{|z|=\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}}\right]
\end{aligned}$$

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Now, if $|a| < \rho$,

$$\int_{|z|=\rho} \frac{dz}{z-a} = \int_{|z-a|=\alpha} \frac{dz}{z-a} = 2\pi i$$

where α is a positive real number less than $\rho - |a|$. Also, not $\frac{\rho^2}{\bar{a}}$ is outside the circle $|z| = \rho$ and thus,

$$n(\gamma, \frac{\rho^2}{\bar{a}}) = \int_{|z|=\rho} \frac{dz}{z-\frac{\rho^2}{\bar{a}}} = 0.$$

Thus,

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{i\rho}{|a|^2-\rho^2} \cdot 2\pi i = \frac{2\pi\rho}{\rho^2-|a|^2}. \checkmark$$

If instead, $|a| > \rho$, we similarly compute

$$\int_{|z|=\rho} \frac{|dz|}{|z-a|^2} = \frac{i\rho}{|a|^2-\rho^2} \cdot -2\pi i = \frac{2\pi\rho}{|a|^2-\rho^2}. \checkmark$$

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