4.3.3 (133): 1,2
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4.3.3

1) Let $f(z) = w = z^2 + z$, and let $g(z) = f(z) + 1/4$. Clearly, f is bijective on a domain D if and only if g is. However,

$$g(z) = (z + \tfrac{1}{2})^2,$$

meaning g maps two poins z and z' to the same point if and only if $1/2 + z = -1/2 - z'$, that is, z and z' are on opposite sides of $-1/2$. In particular, this means that the radius r of our circle around the origin must be less than $1/2$, since otherwise the domain of g will contain an open neighborhood around $-1/2$. However, if $r \leq 1/2$, then for each point $|z| < r$, $\mathrm{Re}(1/2 + z) > 0$ and $\mathrm{Re}(-1/2 - z) < 0$. Thus, g and therefore f does not repeat any values on $|z| < 1/2$, ~~so~~ this is the largest disk around the origin for which the given mapping is one to one.

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2) Let $z_1 = x + iy$ and $z_2 = a + bi$ for $a, b, x, y \in \mathbb{R}$. Observe that

$$\begin{aligned} & e^{z_1} = e^{z_2} \\ \Leftrightarrow\ & e^x(\cos(y) + i\sin(y)) = e^a(\cos(b) + i\sin(b)). \end{aligned}$$

Since the magnitudes must be equal, we know $x = a$. Furthermore, if $\cos(y) = \cos(b)$ and $\sin(y) = \sin(b)$, then necessarily $y = 2\pi k + b$ for some $k \in \mathbb{Z}$. Thus, we will certainly have repeated values if the radius r of the circle around the origin is more than π, however we cannot have repeated values if $r \leq \pi$. Thus, $r = \pi$ is the largest value for which $w = e^z$ is one to one on the circle $|z| < \pi$.

4.3.4

1) Observe that by (36), for any $|z| < 1$ and $|z_0| < 1$, we have

$$\begin{aligned} & \frac{|f(z) - f(z_0)|}{1 - \overline{f(z_0)}f(z)} \leq \frac{|z - z_0|}{1 - \overline{z_0}z} \\ \Rightarrow\ & \frac{|f(z) - f(z_0)|}{|z - z_0|} \leq \frac{1 - \overline{f(z_0)}f(z)}{1 - \overline{z_0}z}. \end{aligned}$$

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As this holds for arbitrary z and z_0, we may take the limit as z goes to z_0, yielding

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2},$$

which is equivalent to the desired result. Furthermore, if $|z| = 1$, then by the convention established in this class, if $|z| = 1$, then $1/(1 - |z|) = 1/0$ is infinite. In this case, the identity trivially holds. $\square$

2)

Define $F(z) : \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\} \to \{z \in \mathbb{C} | \mathrm{Im}(z) \geq 0\}$ as

$$F(z) = \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \frac{z - \overline{z_0}}{z - z_0}.$$

It is clear from this expression that $|F(z)| \to 1$ as $\mathrm{Im}(z) \to 0$ of $|z| \to \infty$ (where z_0 is fixed). We can rearrange the identity to get

$$\left| \frac{\frac{f(z)-f(z_0)}{z-z_0}}{f(z) - \overline{f(z_0)}} \right| \leq \left| \frac{1}{z - \overline{z_0}} \right|$$

As f is differentiable, F has a removable singularity at z_0, so by a trivial continuation we can assume F is analytic on the upper half plane. Thus, we have an analytic function satisfying

$$\limsup_{z \to a} |F(z)| \leq 1$$

on all points $a \in \partial(\{z \in \mathbb{C} | \mathrm{Im}(z) > 0\})$. By the maximum modulus principle, we have for all z in our domain

$$\begin{aligned} |F(z)| &\leq 1 \\ \Leftrightarrow \left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \frac{z - \overline{z_0}}{z - z_0} \right| &\leq 1 \\ \Leftrightarrow \left| \frac{f(z) - f(z_0)}{f(z) - \overline{f(z_0)}} \right| &\leq \left| \frac{z - z_0}{z - \overline{z_0}} \right|, \end{aligned}$$

As desired. Taking the limit as z_0 approaches z, the expression becomes

$$\frac{|f'(z)|}{|2\mathrm{Im}\ f(z)|} \leq \frac{1}{|2y|}.$$

By our restriction $y > 0$, $\mathrm{Im}\ f(z) \geq 0$, we can say

$$\frac{|f'(z)|}{\mathrm{Im}\ f(z)} \leq \frac{1}{y},$$

as desired. $\square$