1. Proof: If f has no zero in Ω, $1/f$ is analytic in Ω and continuous on its boundary. According to Maximum Modulus Principle. $|\frac{1}{f}|$ attains its maximum on $\partial\Omega$,

which means $|f|$ attains its ~~maximum~~ minimum on $\partial\Omega$

If f has a zero in Ω. there is nothing to prove since $|f|$ will attain the minimum on that zero.

2. Proof: If f has no zero in Ω, $1/f$ is analytic in Ω and continuous on its boundary. According to Maximum Modulus Principle $|\frac{1}{f}|$ attains its maximum on $\partial\Omega$

Since $|f| = c$ on the boundary of Ω

$|\frac{1}{f}| = \frac{1}{c}$ on $\partial\Omega$, and $|\frac{1}{f}| \leq \frac{1}{c}$ for $z \in \Omega$. that's

$|f| \geq c$

By the same way, for $f(z)$, we get $|f| \leq c$ for $\forall z \in \Omega$

Therefore $|f| = c$ for $\forall z \in \Omega$

Since f is analytic and bounded. according to Liouville's thm. f must be a constant

If f has a zero in Ω, there is nothing to prove

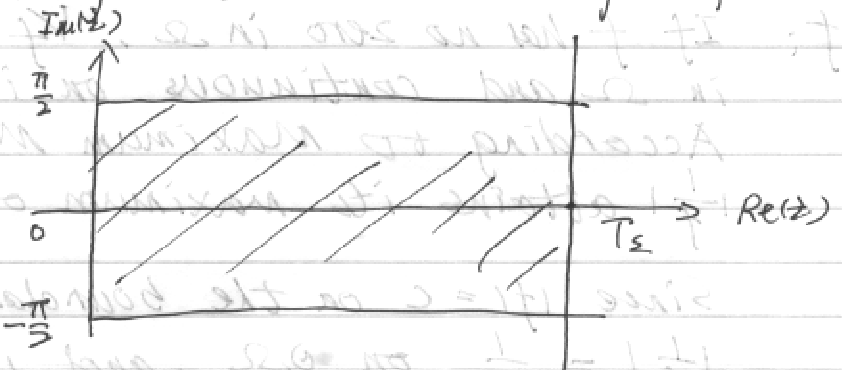
3. Proof: For $c \in (0,1)$, there exists another constant D, s.t. $c < D < 1$.

For $\forall \varepsilon > 0$, the function

$F_\varepsilon(z) = \frac{f(z)}{e^{\varepsilon e^{Dz}}}$ is bounded by 1 on the edges of the half-strip.

And $F_\varepsilon(z) \to 0$ ~~for $\forall z \in \Omega$~~ uniformly for $Im(z)$ as $Re(z) \to \infty$, $z \in \Omega$

Thus we can find a $T(\varepsilon)$ which is dependent on ε, s.t. $|F_\varepsilon(z)|$ is bounded by 1 on the edge of a rectangle



$0 \le Re(z) \le T(\varepsilon)$, $-\frac{\pi}{2} \le Im(z) \le \frac{\pi}{2}$

According to Maximum Modulus Principle $|F_\varepsilon(z)| \le 1$ on the interior of the rectangle.

That's $|f(z)| \le e^{\varepsilon e^{Dz}}$

Let $\varepsilon \to 0$, then $T(\varepsilon) \to \infty$, and $|f(z)| \le 1$

4. Proof: If γ is homotopic to γ_a, there exists a continuous function $\Gamma_a(s,t)$, s.t.

$\Gamma_a(s,0) = \gamma_a(s) \equiv a$

$\Gamma_a(s,1) = \gamma$

for $s \in [0,1]$, $t \in [0,1]$, and

$\Gamma_a(0,t) = \Gamma_a(1,t)$

Let $\Gamma_b(s,t) = \Gamma_a(s,t) + (1-t)(b-a)$

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We can check

$\Gamma_b(s,0) = \Gamma_a(s,0) + b - a \equiv b$

$\Gamma_b(s,1) = \Gamma_a(s,1) + 0 = \gamma$

$\Gamma_b(0,t) = \Gamma_a(0,t) + (1-t)(b-a)$

$= \Gamma_a(1,t) + (1-t)(b-a)$

$= \Gamma_b(1,t)$

Therefore $\Gamma_b(s,t)$ defines a continuous ~~[illegible]~~ deformation from $\gamma_b(t) \equiv b$ to γ so γ is homotopic to the constant curve ~~[illegible]~~ $\gamma_b(t) \equiv b$ for $\forall b \in \Omega$

5. Proof:

~~Let $f(z) = \frac{1}{z}$.~~

~~[illegible]~~

~~[illegible]~~

~~For a closed curve $\gamma_1(\theta) = e^{i\theta}$ $(\theta \in [0, 2\pi])$~~

~~define a deformation $\cancel{\Gamma(s,\theta)}$~~

~~$\Gamma(\theta, t) = \frac{1}{1+t} e^{i\frac{\theta}{1+t}}$, then~~

~~$\Gamma(\theta, 0) = e^{i\theta} = \gamma_1(\theta)$~~

~~$\Gamma(\theta, 1) = \frac{1}{2} e^{i\frac{\theta}{2}}$~~

Let $f(z) = \frac{1}{z}$

For a closed curve $\gamma_1(\theta) = e^{i\theta}$ $(\theta \in [0, 2\pi])$ defines a deformation $\Gamma(\frac{\theta}{2\pi}, t)$, s.t. $\Gamma(\frac{\theta}{2\pi}, t) = (1+t) e^{i\frac{\theta}{1+t}}$, then

$\Gamma(\theta, 0) = e^{i\theta} = \gamma_1(\theta)$

$\Gamma(\theta, 1) = 2 e^{i\frac{\theta}{2}} = \gamma_2(\theta)$

Then $\gamma_2(\theta)$ is homotopic to $\gamma_1(\theta)$.

but $\gamma_2(\theta)$ is not closed, since $\gamma_2(0) \neq \gamma_2(2\pi)$.

Therefore

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_1} \frac{dz}{z} = \int_0^{2\pi} i\,d\theta = 2\pi i$$

$$\int_{\gamma_2} f(z)\,dz = \int_{\gamma_2} \frac{dz}{z} = \int_0^{\pi} \frac{i}{2}\,d\theta = \frac{\pi i}{2}$$

$$\int_{\gamma_1} f(z)\,dz \neq \int_{\gamma_2} f(z)\,dz$$