On Friday we proved the following:

Theorem: Ω: open, connected, f continuous on Ω. Then f has an analytic antideriv. F on Ω $\iff$ $\displaystyle\int_\gamma f\,dz = 0$ for all smooth, closed paths $\gamma \subseteq \Omega$.

Trying to use this to prove:

Cauchy's Thm for Rectangles: R: rectangle. ∂R: boundary of R, f: analytic on R. Then $\displaystyle\int_{\partial R} f\,dz = 0$.

pf idea: f analytic $\Rightarrow$ at any pt z^*, secants through z^* closely approx. tangents:

$$\text{if } |z - z^*| < \delta \Rightarrow \left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon$$

$$\iff \Big| f(z) - \big(\underbrace{f(z^*) + (z-z^*)f'(z^*)}_{\substack{\text{tangent line @ } z^* \\ T_{z^*}(z)}} \big) \Big| < \epsilon\,|z - z^*| \qquad (1)$$

But tangent line has antiderivative that is parabola, so analytic.

By previous thm. $\Rightarrow$ $\displaystyle\int_\gamma T_{z^*}(z)\,dz = 0$ for closed, smooth γ in R.

if we take $\gamma \subseteq B(z^*, \delta)$, then the estimate (1) applies, so

$$\left| \int_\gamma f\,dz \right| = \left| \int_\gamma \big(f(z) - T_{z^*}(z)\big)\,dz \right| < \varepsilon \int_\gamma |z - z^*|\,|dz|$$

Plan: Chop R into pieces, express $\int_{\partial R} f\,dz$ in terms of pieces

Take piece whose line integral is large and estimate it using our prior work.

See notes from previous lecture for details.

Combining previous two theorems, we obtain much more general result:

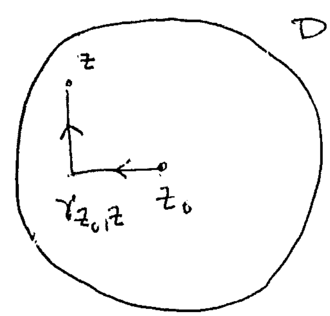
Cauchy's Thm in a Disk: f analytic on D, then $\int_\gamma f(z)\,dz = 0$

for any closed curve $\gamma \subseteq D$ (smooth)

(Note: Not just ∂D as in previous theorem.)

pf: By earlier theorem, suffices to show $\exists$ F: analytic on D such that $\frac{d}{dz} F = f$ (i.e. $f\,dz$ "exact")

Play a similar game: $F(z) \overset{\text{def}}{=} \int_{\gamma_{z_0,z}} f\,dz$ $\quad$ $\gamma_{z_0,z}$ rect. path with two segments as in picture: (z_0: center of D)



(Choice of path does matter here!
In pf. of prev. theorem, we had assumed indep. of path.)

Approaching along $\gamma_{z_0,z}$, clear that

$$\frac{\partial F}{\partial y} = i f$$

(same reasoning as in earlier pf.)
with $f\,dz = f\,dx + i f\,dy$

Remains to show $\frac{\partial F}{\partial x} = f$. For this, note that taking $\sigma_{z_0,z}$ two-segment rect. path forming rectangle R with $\gamma_{z_0,z}$, then

since $\int_{\partial R} f\,dz = 0$ by previous thm, $\Rightarrow \underbrace{\int_{\gamma_{z_0,z}} f\,dz}_{\| \\ F(z)} = \int_{\sigma_{z_0,z}} f\,dz$

Using the path $\sigma_{z_0,z}$, it is clear that

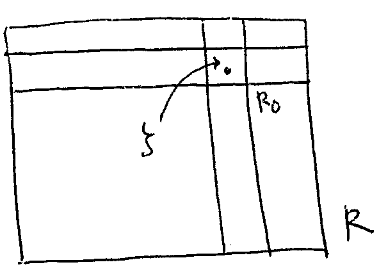
$\frac{\partial F}{\partial x} = f \quad \Rightarrow \quad F$ analytic at z with $\frac{dF}{dz} = f$. Since z arbitrary, we're done. //

Suppose instead f analytic on, for starters, $R - \{\xi\}$ ← single point inside rectangle.

Can we conclude anything about $\int_{\partial R} f\,dz$?

Yes, with some assumptions.

(clearly not always 0, as example $\int_{\partial R} \frac{1}{z}\,dz$ would show)



Cut R into 9 pieces, one of which R_0 contains ξ (might as well assume R_0 square, clear that it can be made arbitrarily small)

Then $\int_{\partial R} f\,dz$ = sum of line ints around 9 pieces, 8 of which are 0 by prev. theorem $= \int_{\partial R_0} f\,dz$.

But to conclude anything about $\int_{\partial R_0} f\, dz$, we need an estimate on f near ξ. (i.e. some limit as $z \mapsto \xi$ of f is small)

If $\lim\limits_{z \to \xi} (z-\xi) f(z) = 0$ then $|f(z)| < \dfrac{\varepsilon}{|z-\xi|}$ for any $\varepsilon > 0$.

$$\Rightarrow \quad \left| \int_{\partial R_0} f\, dz \right| < \varepsilon \cdot \int_{\partial R_0} \frac{|dz|}{|z-\xi|} . \qquad (*)$$

If R_0 has side length L, then $|z-\xi| \geq \frac{1}{2} L$ (since ξ is center of R_0)

$$\text{So} \quad (*) \leq \frac{\varepsilon \cdot 2}{L} \int_{\partial R_0} |dz| = \frac{\varepsilon \cdot 2}{L} \cdot 4L = 8\varepsilon .$$

Since ε arbitrary $\Rightarrow$ $\displaystyle\int_{\partial R} f\, dz = \int_{\partial R_0} f\, dz = 0.$

Thus we've proved:

<u>Theorem</u>: f analytic on $R' = R - \{\xi_j\}_{j=1}^{k}$ s.t.

$$\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0 \quad \forall\ j = 1, \dots, k, \quad \text{then} \int_{\partial R} f\, dz = 0.$$

Or combining with Cauchy's thm. on disk:

<u>Thm</u>: f analytic on $D' = D - \{\xi_j\}_{j=1}^{k}$ with limit condition,

then $\displaystyle\int_{\gamma} f\, dz = 0$ for all smooth closed paths $\gamma \subseteq D'$.

(same proof with very slight modification, as γ must not contain ξ_j).