Theorem: $f$ analytic on $D - \{\xi_j\}_{j=1}^{k}$ such that

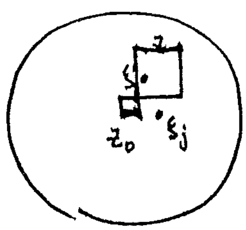
$$\lim_{z \to \xi_j} (z - \xi_j) f(z) = 0 \quad \forall\ j = 1, \ldots, k, \quad \text{then}$$

$$\int_\gamma f(z)\,dz = 0$$ for any (smooth) closed curve $\gamma \subseteq D - \{\xi_j\}$.

---

pf: combine earlier results. Proved true on disk $D$, true for a rectangle $R - \{\xi_j\}_{j=1}^{k}$ for finite # of pts. $\xi_j$

Just choose rectilinear paths in $D$ to avoid $\xi_j$'s:




Complete proof as before.

---

Key ingredient in pf. of Cauchy's integral formula: Suppose $f$ analytic on $D$.
$\gamma$: closed curve in $D$
$a$: pt. in $D - \gamma$.

Let $$\phi(z) = \frac{f(z) - f(a)}{z - a}$$

Then $$\lim_{z \to a} \phi(z) \cdot (z - a) = \lim_{z \to a} f(z) - f(a) = 0$$ (since $f$ analytic $\Rightarrow$ $f$ continuous)

Hence: $$\int_\gamma \left( \frac{f(z) - f(a)}{z - a} \right) dz = 0.$$

or equivalently: $$\int_\gamma \frac{f(z)}{z - a}\,dz = f(a) \cdot \int_\gamma \frac{1}{z-a}\,dz$$

circle centered at $a$ with suff. small radius $r$

If $\gamma = C(a; r) \subseteq D$, then $$\int_\gamma \frac{1}{z-a}\,dz = 2\pi i$$

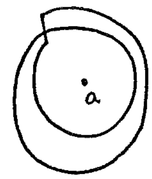
Conclusion : $$f(a) = \frac{1}{2\pi i} \int_{C(a;r)} \frac{f(z)}{z-a}\, dz$$

We can understand the value of the (analytic) function at any point $a$ in disk, just by knowing its values on $C(a;r)$. In fact, similar statement is true for general closed curves in $\gamma$. Just need to analyze $\displaystyle\int_{\gamma} \frac{dz}{z-a}$

Believe the circle $C(a;r)$ : centered at $a$ radius $r$ is fundamental.

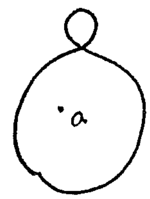
Traversing circle twice counter-clockwise gives $2\cdot(2\pi i)$. Expect

a curve of shape:



to give similar answer.

Further, if $\gamma$ has self-intersections :



then can be separated into multiple

piece-wise smooth closed curves $\gamma_i$. Previous thm $\Rightarrow$ $\int_{\gamma_i} \frac{1}{z-a}\,dz = 0$ if $\gamma_i$ does not contain $a$.

(since $\frac{1}{z-a}$ analytic on interior of $\gamma_i$ or disk containing $\gamma_i$ not $a$.)

<u>Conjecture</u> :
$$\int_\gamma \frac{dz}{z-a} = \text{integer multiple of } 2\pi i$$

if $\gamma$ closed curve containing $a$.

How to prove this? $2\pi i$ is period for $e^z$. If

$\gamma$ parametrized by $z(t)$, $t \in [\alpha, \beta]$, then

$$\int_\gamma \frac{dz}{z-a} = \int_\alpha^\beta \frac{z'(t)}{z(t)-a}\,dt$$
so consider

$$h(t) := \int_\alpha^t \frac{z'(s)}{z(s)-a}\,ds ,$$
continuous, $\mathbb{R} \to \mathbb{C}$

with $h'(t) = \dfrac{z'(t)}{z(t)-a}$. Want to show $e^{h(\beta)} = 1$.

$$\frac{d}{dt} e^{h(t)} = \frac{z'(t)}{z(t)-a} \cdot e^{h(t)}, \quad \text{so} \quad \frac{d}{dt}\left(z(t)-a\right)e^{-h(t)} = 0$$

for all $t$.

$\Rightarrow$ $(z(t)-a)\,e^{-h(t)}$ is constant.

(if $\gamma$ only assumed piecewise smooth, then $h'(t)$ defined for all but finitely many $t \in [\alpha, \beta]$ $\Rightarrow$ derivative $= 0$ for all but finitely many $t$.)

if $t = \alpha$, then $h(\alpha) = 0$, so

$$z(\alpha) - a = (z(t)-a)\,e^{-h(t)}$$

i.e. $$e^{h(t)} = \frac{z(t)-a}{z(\alpha)-a}.$$

Setting $t = \beta$, we see that the right-hand side is $1$. So $e^{h(\beta)} = 1$

$\Rightarrow$ $h(\beta) = \displaystyle\int_\alpha^\beta \frac{dz}{z(t)-a}$ is multiple of $2\pi i$.

Define "winding number" $n(\gamma, a)$ to be this integer: $\quad n(\gamma, a) = \displaystyle\int_\gamma \frac{dz}{z-a} \cdot \left(\frac{1}{2\pi i}\right)$

Properties: ① $n(\gamma, a) = -n(-\gamma, a)$

② $n(\gamma, a) = 0$ if $\gamma$ contained in disk $D$, $a \notin D$.

③ $\gamma$ cuts $\mathbb{C} \cup \{\infty\}$ into open, connected sets "regions". $n(\gamma, a)$ constant on a region.

$n(\gamma, a) = 0$ on region with $\{\infty\}$.

pf of 3 : Any two points $a, b \in \Omega$ : open connected

are joined by path consisting of straight-line segments.

So suffices to examine case when $a, b$ joined by a single straight line segment.

show $n(\gamma, a) = n(\gamma, b)$ in this case.

Clever fact : $\frac{z-a}{z-b}$ is only real, negative on segment connecting $a$ to $b$.

$\Rightarrow$ $\log\left(\frac{z-a}{z-b}\right)$ is well-defined single valued function off the line segment $[a, b]$.

with derivative $\frac{1}{z-a} - \frac{1}{z-b}$, so $\int_\gamma \left(\frac{1}{z-a} - \frac{1}{z-b}\right) dz = 0$ for any pts $a, b$ in interior of component defined by $\gamma$.

i.e. $n(\gamma, a) = n(\gamma, b)$.

To show $n(\gamma, a) = 0$ if $a$ in component containing $\infty$, already know constant, so

just need to know $n(\gamma, a) = 0$ at some point.

Pick a with $|a|$ suff. large so that $\gamma$ contained in a disk away from a.

Thus we have proved the following theorem:

Theorem: Suppose $f$ is analytic on open disk: $D$, $\gamma$ closed curve in $D$. then for any point $a \notin \gamma$, then

$$f(a) = \frac{1}{2\pi i \cdot n(\gamma, a)} \int_{\gamma} \frac{f(z)\,dz}{z-a}$$

$n(\gamma, a)$: winding number.