On Wednesday, proved Cauchy's integral formula:

Given f: analytic on disk D, $\gamma \subseteq D$, a: pt. not on γ

then $$f(a) = \frac{1}{2\pi i\, n(\gamma,a)} \cdot \int_\gamma \frac{f(z)\,dz}{z-a}$$ $n(\gamma,a)$: winding # of γ about a.

pf used Cauchy's thm on disk $-$ $\{$finite # of pts $\xi_j\}$, applied to

$$\phi(z) := \frac{f(z) - f(a)}{z-a}.$$

Cauchy's int. formula gives us explicit expressions for derivatives as well.

Rewrite integral formula:

$$f(z) = \frac{1}{2\pi i\, n(\gamma,z)} \int_\gamma \frac{f(\xi)\,d\xi}{\xi - z}$$

Then can hope that

$$f'(z) = \frac{1}{2\pi i\, n(\gamma,z)} \int_\gamma \frac{f(\xi)\,d\xi}{(\xi-z)^2}$$ (by differentiating integrand w.r.t. parameter z.)

Indeed if

$\phi(t,z) : [a,b] \times \Omega \longrightarrow \mathbb{C}$ $\quad \Omega$: open, ϕ continuous

and analytic as a function of z for fixed t, with $\frac{\partial \phi}{\partial z}$ continuous (i.e. cont. first partials)

then $\int_a^b \phi(t,z)\,dt =: \Phi(z)$ is

analytic on Ω with derivative $\Phi'(z) = \int_a^b \frac{\partial \phi}{\partial z}\,dt$

The proof can be reduced to the real case:

$\phi(t, x) : [a,b] \times [c,d]$ continuous. ... same statement.

Estimate difference quotient for $\Phi(x)$ via mean value thm.

Then use uniform continuity. //

Ahlfors has elegant, direct argument for showing one can differentiate w.r.t. parameter under integral sign for just the kind of functions that appear in the Cauchy integral formula.

Lemma : $\phi(\zeta)$: continuous on arc γ. Then

$$F_n(z) = \int_\gamma \frac{\phi(\xi)}{(\xi - z)^n}\, d\xi$$

defines analytic function in each of the regions determined by γ and

$$F_n'(z) = n \cdot F_{n+1}(z)$$

(By induction)

pf: F_1 continuous :

Given $z_0 \notin \gamma$, find open nbhd $B(z_0, \delta)$ such that $B(z_0,\delta) \cap \gamma = \phi$. i.e. $|\xi - z| > \delta$ if $\xi \in \gamma$.

(δ as small as we like)

Now $F_1(z) - F_1(z_0) =$

$$(z - z_0) \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z)(\xi - z_0)}$$

upon writing

$$\frac{1}{(\xi - z)} - \frac{1}{(\xi - z_0)} = \frac{z - z_0}{(\xi - z)(\xi - z_0)}$$

$$\Rightarrow \quad |F_1(z) - F_1(z_0)| < |z - z_0| \cdot \frac{1}{\delta^2} \underbrace{\int_\gamma |\phi(\xi)|\, |d\xi|}_{\text{fixed const. indep. of } z, z_0}$$

so F_1 continuous as desired.

moreover

$$\frac{F_1(z) - F_1(z_0)}{z - z_0} = \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z)(\xi - z_0)} .$$

As $z \to z_0$, RHS tends to

$$\int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z_0)^2} = F_2(z_0)$$

(as left-hand side is continuous function)

i.e. $F_1'(z_0) = F_2(z_0)$

for all points z_0 in open, connected sets defined by γ.

This is base case of induction.

(4)

Suppose now $F_{n-1}'(z) = (n-1) F_n(z)$. We have the slightly messier identity:

$$F_n(z) - F_n(z_0) = \left[\int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z)^{n-1}(\xi - z_0)} - \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z_0)^n} \right] +$$

$$(z - z_0) \cdot \int_\gamma \frac{\phi(\xi)\, d\xi}{(\xi - z)^n (\xi - z_0)}$$

from $\frac{1}{(\xi - z)^n}$ ~~$\frac{1}{(\xi-z_0)^n}$~~

$$= \frac{1}{(\xi - z)^{n-1}} \frac{(\xi - z_0)}{(\xi - z_0)} \qquad \frac{z - z_0}{\xi - z_0} + \frac{\xi - z}{\xi - z_0} = \frac{\xi - z_0}{\xi - z_0} = 1.$$

$\Rightarrow$ F_n continuous, since by inductive hypothesis applied to $\frac{\phi(\xi)}{\xi - z_0}$, the integrals in brackets are continuous.

The latter integral can be estimated as before, in abs. value: $< |z - z_0| \frac{1}{\delta^{n+1}} \int_\gamma |\phi|$

Similar to before, also divide both sides of identity by $z - z_0$.

Take limit as $z \to z_0$.

By inductive hypothesis get the term in brackets giving $(n-1) F_{n+1}(z_0)$

since we apply inductive hypothesis to $\tilde{F}_{n-1}(z) = \int_\gamma \frac{\phi(\xi)/(\xi - z_0)\, d\xi}{(\xi - z)^{n-1}}$

while latter term gives $F_{n+1}(z_0)$

for total of $n \cdot F_{n+1}(z_0)$.

Apply this to integrals appearing in Cauchy Integral formula:

e.g. γ = circle C (traversed once, so winding # is one)

then $$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} F_1(z)$$

so $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^{n+1}},$$ an analytic function on interior / exterior of circle C.

Play with this in various ways:

f defined, continuous on Ω: open, conn., $\int_\gamma f\, dz = 0$ for all closed curves $\gamma \subseteq \Omega$

$\Rightarrow$ (earlier thm) f is derivative of analytic function F

$\Rightarrow$ (using Cauchy integral thm. on F and diff.) f is analytic. (Morera's thm)

Liouville's thm: Function ~~which~~ which is analytic + bounded in whole plane must be constant.

pf: Suppose C: circle of radius r, $|f(\xi)| \le M$ on C

then by formula for n^{th} derivative: $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\xi)\, d\xi}{(\xi - z)^{n+1}}$$

Then for any pt. $z=a$ not on circle:

$$\left| f^{(n)}(a) \right| \le M \cdot n!\, r^{-n} \qquad \text{(Cauchy's estimate)}$$

nevermind the $2\pi i$.

Use this for $n=1$: our hypothesis that f bounded means $\exists\, M$ s.t.

$|f(\xi)| \le M$ on any circle C ~~on~~, say containing a,

hence Cauchy's estimate $\Rightarrow$ $|f'(a)| = 0$ by taking $r \to \infty$.

(ie. $f'(a)=0$)

need fact that f defined on whole complex plane to do this.

$\Rightarrow$ f constant.

<u>Cor</u>: (Fundamental Thm. of Algebra)

Every polynomial, $p(z)$, ~~[illegible]~~ has a complex root.

<u>pf</u>: Suppose not. Then $1/p(z)$ is analytic,

and bounded since $\left|\frac{1}{p(z)}\right| \to 0$ as $z \to \infty$.

$|\cdot| : \mathbb{C} \cup \{\infty\} \to \mathbb{R}$ is continuous function, so attains maximum.

But then Liouville's thm implies that $1/p(z)$ constant. ↯.