Last time, defined residue of isolated singularity a_j

as $R_j(f) = \frac{1}{2\pi i} \int_{C(a_j;\delta)} f(z)\,dz$

$C(a_j;\delta)$: small circle of radius δ s.t. f analytic for $0 < |z-a_j| \leq \delta$.

characterized R_j as the unique cx. number

s.t. $f(z) - \frac{R_j}{z-a_j}$ is derivative of an analytic function on nbhd. of a_j.

(could use any smooth, closed curve γ with $n(\gamma;\delta)=1$)

(Reason: for any path γ (smooth, closed) $\subseteq B(a_j;\delta) \setminus \{a_j\}$

then $\gamma \approx n(\gamma;a_j)\cdot C(a_j;\delta')$. $\delta' < \delta$.)

but since $\int_{n(\gamma;a_j)C(a_j;\delta)} \left(f(z) - \frac{R_j}{z-a_j} \right) dz = 0$ by construction, then

use thm about having antiderivative in Ω $\Longleftrightarrow$ ints. over closed curves γ in Ω are 0.

What does "$\sim$" mean? Homologous to.

i.e. $\gamma \approx n(\gamma;a_j)\, C(a_j;\delta')$ means

$\gamma - (n(\gamma;a_j)\, C(a_j;\delta')) \approx 0$ means winding number of all points outside of $B(a_j;\delta) \setminus \{a_j\}$ are 0.

(could interpret via homotopy. Less general. Maybe less good.)

If we choose different cx. number, say S_j, then clear that choosing $\gamma = C(a_j;\delta)$ itself, $\int_{C(a_j;\delta)} \left(f - \frac{S_j}{z-a_j}\right) dz \neq 0 \Rightarrow$ uniqueness.

If we set $R_j(f) = \dfrac{P_j(f)}{2\pi i}$ then result, called Residue of f at a_j,

is unique cx. # such that $f(z) - \dfrac{R_j}{z-a_j}$ is derivative of

(single-valued) analytic function on suffic. small nbhd of a_j.

Given any $\gamma \subseteq \Omega$ with $\gamma \sim 0$, then in the set $\Omega \setminus \{a_1, \dots, a_n\}$

$$\gamma \sim \sum_j n(\gamma, a_j) \cdot C_j \qquad C_j := C(a_j, \delta) \text{ : cycle with winding \# 1 around } a_j.$$

where $n(\gamma, a_j)$ is the winding # in $\Omega \setminus \{a_1, \dots, a_n\}$.

So then in this region:

$$\int_\gamma f\,dz = \int_{\sum_j n(\gamma,a_j) C_j} f\,dz = \sum_j n(\gamma, a_j)\, P_j$$

or, using residues $R_j = P_j/2\pi i$:

$$\frac{1}{2\pi i} \int_\gamma f\,dz = \sum_j n(\gamma, a_j)\, R_j$$

Known as "Residue Thm". Note it applies equally well if f has infinitely many isolated zeros. Indeed, we just need to show that for any γ, $n(\gamma, a_j) = 0$ for all but fin. many j. This is so because $n(\gamma, a) = 0$ for any a in unbounded component of a. (i.e. other components bounded, say inside disk of radius R, for $R \gg 0$.) But then isolated singularities in this disk must be finite - else they would have an accumulation pt.

Precise statement: f analytic except for isolated singularities in Ω. $\{a_j\}$

$\gamma \subseteq \Omega$ with $\gamma \sim 0$, and none of $a_j \in \gamma$. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z)$$

Notice that, as we have chosen to define it, the residue thm is easy to prove. but it isn't clear how to calculate residues.

If a: pole, then in a nbhd of a, f has the expansion

$$f(z) = \frac{a_{-h}}{(z-a)^h} + \cdots + \frac{a_{-1}}{(z-a)} + \underbrace{\phi(z)}_{\substack{\text{analytic} \\ \text{in nbhd of } a}} \qquad \text{if } a \text{ is a pole of order } h$$

Then $\operatorname{Res}_{z=a} f(z) = a_{-1}$, since ϕ and other monomials have antiderivatives defined in nbhd of a.

Multiplying by $(z-a)^h$, we see that

$$a_{-1} = \frac{1}{(h-1)!}\frac{d^{h-1}}{dz^{h-1}}\left((z-a)^h f(z)\right)\Bigg|_{z=a}$$

In particular if $h=1$: (that is only if pole is of order 1!)

$$a_{-1} = \lim_{z\to a} (z-a) f(z)$$

In fact, same is true for essential singularities, like $e^{1/z}$ at $z=0$, using Laurent series which we now present.

Before dealing with Laurent series, let's discuss power series.

previously noted that, given f analytic in disk centered at a,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}\, d\xi$$

, via C.I.F. + diff. under integral sign.

so idea: for z close to a (i.e. $|z-a| < r < R$ where f analytic on $B(a,R)$)

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi-a)^{n+1}}\, d\xi \cdot (z-a)^n$$

start from C.I.F.: $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)\, d\xi}{(\xi - z)}$ $\gamma = a + re^{it}$, $t \in [0, 2\pi]$

Now $$\frac{1}{(\xi - z)} = \frac{1}{(\xi-a)} \cdot \underbrace{\frac{1}{1-(z-a)/(\xi-a)}}_{\text{geom. series}} = \frac{1}{(\xi-a)} \cdot \sum_{n=0}^{\infty} \frac{(z-a)^n}{(\xi-a)^n}$$

on γ, (i.e. for $\xi \in \{\gamma\}$) $\left|\frac{f(\xi)}{(\xi-a)} \cdot \frac{(z-a)^n}{(\xi-a)^n}\right| \leq \frac{M}{r} \cdot \underbrace{\frac{|z-a|^n}{r^n}}_{<1 \text{ if } z \in B(a,r)}$ for some M since f attains max on γ.

$\Rightarrow$ series made from summands:

$$\sum_{n=0}^{\infty} \frac{f(\xi)}{(\xi-a)^{n+1}} (z-a)^n$$

converges uniformly for $\xi \in \{\gamma\}$ by Weierstrass M-test. (comparison with geometric series)

if $F_n \xrightarrow{\text{unif. on } \gamma} F$ then $\int_\gamma F = \int_\gamma F_n$. ✓

and $r < R$ arbitrary so have convergence for all $|z-a| < R$