Residue thm: $f$ has isolated singularities $\{a_j\}$ in $\Omega$, $\gamma \approx 0$ in $\Omega$

$$\frac{1}{2\pi i}\int_\gamma f\, dz = \sum_j n(\gamma, a_j)\cdot \mathrm{Res}_{z=a_j}(f)$$

view this as simultaneous generalization of both C.I.F. and Cauchy's thm.

C.I.F. $\frac{f(z)}{z-a} \rightsquigarrow \mathrm{Res}_{z=a} = f(a)$.

Cauchy's thm. $f$ with no poles $\rightsquigarrow$ no residues, i.e. $\sum = 0$.

—

Have Laurent expansion for $f$ analytic in annulus $\mathrm{Ann}(z_0, R_1, R_2)$.

For isolated singularity at $z_0$, consider $\mathrm{Ann}(z_0, 0, R)$ for some $R$

Is it still true that the residue at $z_0$ is the $-1^{th}$ coeff. of Laurent series at $z=a_j$?

Yes! Clear that $f(z) - \frac{a_{-1}}{z-z_0}$ is derivative of a single-valued function on $0 < |z-z_0| < R$

if we can differentiate power series / Laurent series term by term.

(again application of uniform convergence on compact subsets of $0<|z-z_0|<R$, which allows us to interchange limits and integration, expressing derivs. via Cauchy integral formula)

Thus $a_{-1}$ must be residue, by our earlier characterization as unique ex. # with this prop.

Residue theorem gives generalization of our result on zeros:

$$\sum_j n(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz$$

$z_j$ : zeros of $f(z)$. $f$ analytic on $\Omega$ with $\gamma \approx 0$.
(none of $z_j$ lie on $\gamma$)

Now suppose $f$ meromorphic, which means (by definition) that $f$ has only isolated singularities, none of which are essential.

If $f$ has a pole of order $h$, so $f(z) = (z-a)^{-h} f_h(z)$ with $f_h(z)$ analytic at $z = a$ (non-zero)

then $f'(z) = (-h)\cdot(z-a)^{-(h+1)} f_h(z) + (z-a)^{-h} f_h'(z)$

$$\Rightarrow \quad f'(z)/f(z) = -h/(z-a) + f_h'(z)/f_h(z)$$

i.e. residue of $f'/f$ at $z = a$ is $-h$.

Thm: $f$: meromorphic on $\Omega$ with zeros at $z_j$, poles at $p_j$ then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'}{f}\, dz = \sum_j n(\gamma, z_j) - \sum_j n(\gamma, p_j)$$

where, as before, it is understood that if $z_j$ or $p_j$ has multiplicity (resp. order) $h$, then it appears $h$ times in the sum.

Of course, this is nicest when $\gamma$ chosen to guarantee all winding numbers are 0 or 1, e.g. circle traversed once counterclockwise.

It is called the "argument principle" because,

$$\int_\gamma \frac{f'}{f}\,dz = \int_{f(\gamma)} \frac{dw}{w}$$

thinking of $w = f(z)$ where $f(\gamma)$ will again be a smooth curve, since $f$ analytic, $\gamma$ smooth.

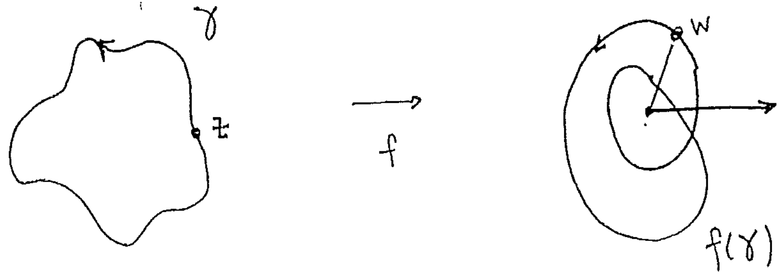
Moreover, since $f$ doesn't vanish on $\gamma$ by assumption, then the curve $f(\gamma)$ avoids origin in $w$-plane, so we may define a continuous (branch of) logarithm $\log(f(\gamma(t)))$

and so

$$\int_{f(\gamma)} \frac{dw}{w} = \log(f(\gamma(t)))\Big|_{t=a}^{t=b}$$

$$= \underbrace{\log|f(z)|\Big|_{z=\gamma(a)}^{z=\gamma(b)}}_{\substack{=0 \text{ since single-valued} \\ \text{and } \gamma \text{ closed}}} + i \arg f(z)\Big|_{z=\gamma(a)}^{z=\gamma(b)}$$

Geometric picture:




e.g. $z^2$, and unit circle.

Corollary (Rouché's Thm) $\gamma \approx 0$ in $\Omega$ s.t. $n(\gamma, z) = 0$ or $1$ for all $z \notin \gamma$. If $f, g$ analytic on $\Omega$ and satisfy

$$|f(z) - g(z)| < |f(z)| \quad \text{on } \gamma$$

then $f(z)$ and $g(z)$ have the same # of zeros enclosed by $\gamma$.

pf : The inequality can be rewritten

$$\left| \frac{g(z)}{f(z)} - 1 \right| < 1$$

$\Rightarrow$ $g/f = F$ has values on $\gamma$ contained in $C(1,1)$

circle centered at $z=1$ with radius 1.

i.e. neither zero, nor pole.

so by Argument principle, ~~right hand side~~ = # of zeros − # of poles of F

$$\int_{F(\gamma)} \frac{dw}{w} = \int_{\gamma} \frac{F'}{F} dz$$

= # of zeros of $g$ − # of zeros of $f$

= 0. (since in w-plane, $F(\gamma)$ trapped away from 0 in $C(1,1)$)

Ahlfors notes that one application is that zeros can be understood often via Taylor expansion : ( say on disk centered at origin w/ radius $R$ )

$$f(z) = \underbrace{P_n(z)}_{n^{th}\text{ Taylor poly.}} + z^{n+1} f_{n+1}(z)$$

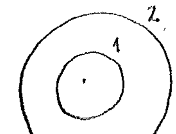
then require $|f(z) - P_n(z)| = \underbrace{|z^{n+1} f_{n+1}(z)|}_{R^{n+1} \cdot |f_{n+1}(z)|} < |P_n(z)|$ on $|z| = R$.

solutions to polynomials can be def'd by approximation

(similar to #2. Spring 2012)

Example of Rouché's Thm. : $2z^5 + 8z - 1 = 0$

Show there are 4 roots in (open) annulus $\text{Ann}(0,1,2)$




$f(z) = 2z^5$ $\qquad \gamma$ : circle of radius 2 $C(0,2)$

$f(z) - g(z) = -(8z-1)$

$g(z) = 2z^5 + 8z - 1$

then $|2z^5| > |8z-1|$ for $|z| = 2$

since $|8z-1| < |8z| + 1 \le 17 < 64$

For $\gamma$ : circle of radius 1,

then roles reversed. Take $f(z) = 8z - 1$. $g(z) = 2z^5 + 8z - 1$

$|2z^5| = 2 < 8 - 1$, so since $f$ has 1 root in $|z| < 1$

then $g(z)$ has 1 such root.

( Note if $|z| = 1$, then $|2z^5 + 8z - 1| > 0$ so remaining 4 roots are in annulus ).

$$\ge \underbrace{|2z^5 + 8z|}_{\substack{2\cdot|z^4+8| \\ \ge 7}} - 1 \ge 13.$$

Qualifying Exam Problem : #7, Fall 2012

$f(z)$ analytic on punctured disk $D \setminus \{0\}$. $\text{Re}(f(z)) > 0$ on this disk.

Show $f(z)$ has removable singularity at 0.

#4, Spring 2012 : No $f$ analytic on $D \setminus \{0\}$. Such that $f'$ has simple pole at 0.