Suppose that $f(z)$ has an isolated singular point at $z = \infty$.

Define the residue at ∞ to be: $\quad \operatorname{Res}_f(\infty) := -\dfrac{1}{2\pi i}\displaystyle\int_\gamma f(z)\,dz$

where γ is any cycle with winding number $= 1$ for all pts in interior of γ and such that all finite singularities are contained in bounded components det'd by γ. E.g. $\gamma = C(0,R)$ with radius $R >> 0$.

Equivalently, we may define $\operatorname{Res}_f(\infty)$ to be $(-a_{-1})$: where a_{-1} is the coefficient of $1/z$ in Laurent expansion of $f(z)$ at ∞.

initially might seem confusing. Show that its right. Conceptual reason later...

Typically, we evaluate this by making change of vars: $\quad z \mapsto 1/z$
$\qquad \infty \mapsto 0$

Then $-\operatorname{Res}_f(\infty)$ is coeff. of z in expansion of $f(\frac{1}{z})$ at $z = 0$.

Or equivalently, coeff. of $\frac{1}{z}$ in Laurent expansion of

$$F(z) := \frac{1}{z^2} f\left(\frac{1}{z}\right) \quad \text{at } z = 0. \qquad (\text{i.e. } \operatorname{Res}_f(\infty) = -\operatorname{Res}_F(0))$$

Indeed, if $\gamma = C(0,R)$, $R >> 0$, then

$$\operatorname{Res}_f(\infty) := -\frac{1}{2\pi i}\int_{C(0,R)} f(z)\,dz = \frac{1}{2\pi i}\int_{-C(0,\frac{1}{R})} f\left(\frac{1}{\xi}\right)\frac{d\xi}{\xi^2}$$

$z \mapsto 1/\xi$
$dz \mapsto -\dfrac{d\xi}{\xi^2}$

$$\frac{1}{2\pi i}\int_{-C(0,\frac{1}{R})} f\left(\frac{1}{\xi}\right)\frac{d\xi}{\xi^2} = -\operatorname{Res}_F(0)$$

Since, if F has simple pole at $\xi = 0$,

then $\operatorname{Res}_F(0) = \lim\limits_{\xi \to 0} \xi \cdot F(\xi)$

so $\operatorname{Res}_f(\infty) = -\lim\limits_{z \to \infty} z \cdot f(z)$ if $f(\infty) = 0$.

(also see this via Laurent series, $f(\infty) = 0$ implies $a_n = 0$ for $n > 0$)

With these definitions:

necessarily if ∞ is sing.
(finitely many)

Thm: If $f(z)$ (as function on Riemann sphere $\mathbb{C} \cup \{\infty\}$) has isolated singularities $\{z_j\}$

then $\displaystyle\sum_j \mathrm{Res}_f(z_j) = 0$.

Residue thm

Pf: defn of $\displaystyle \mathrm{Res}_f(\infty) = -\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = -\sum_{z_j \neq \infty} \mathrm{Res}_f(z_j)$ //

γ contains all finite isolated singularities

Moreover, **Thm**: γ : ~~[illegible]~~ simple closed curve

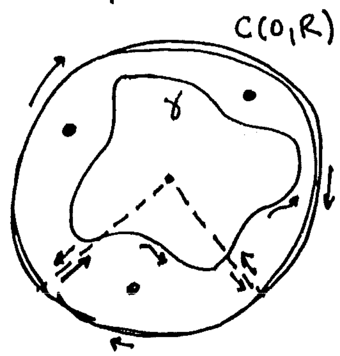
f: analytic on closure of ext. of γ with isolated sings. $\{z_j\}$

$$\int_{\gamma} f\,dz = 2\pi i \sum_j \mathrm{Res}_f(z_j) \qquad (\text{including } z_j = \infty)$$

traverse γ clockwise

further assume winding # $\neq 0,1$ for all points in $\mathbb{C}$

sketch in pictures: Choose $R > 0$ to contain all finite singularities outside γ.



$$\int_{\gamma} f\,dz = \underbrace{\int_{C_1} + \int_{C_2} f\,dz}_{} = -\underbrace{\int_{C(0,R)}}_{} f\,dz + 2\pi i \sum_{z_j \text{ finite}} \mathrm{Res}_f(z_j)$$

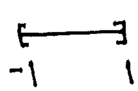
traversed counter clock giving $2\pi i\, \mathrm{Res}_f(\infty)$.

chops region between $C(0,R)$ and γ into closed curves C_1, C_2.

Use this latter thm to study functions with branch points in $|z| < \infty$ and an isolated singularity at $z = \infty$.

E.g. $\sqrt{1-z^2}$ single valued on cut plane:

positive on positive imaginary axis.



To show isolated sing. at ∞:

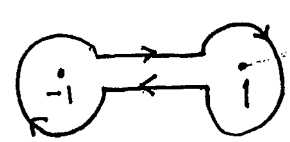
$$\sqrt{1-z^2} = -iz\sqrt{1 - 1/z^2}$$

$-i$ not $+i$ so that posi in iy.

use power series with $\sqrt{1-\xi}$. for $|\xi| <$

Show: $\displaystyle\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx = \pi\left(\sqrt{2} - 1\right)$

Contour:



The power series expansion shows that $\sqrt{1-z^2}$ analytic for $|z|>1$.

$\Rightarrow$ Singularity at infinity is isolated.

Alternate geometric approach to ~~[illegible]~~ problem: is branch well-defined at ∞?

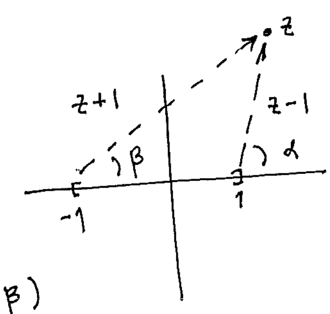
$$\sqrt{1-z^2} = i\sqrt{z^2-1}.$$ Focus on $\sqrt{z^2-1}$ which behaves like z in limit as $|z| \to \infty$

$$\sqrt{z^2-1} = \sqrt{(z-1)(z+1)}.$$

$$\sqrt{z} = z^{1/2} = e^{1/2 \log z} = e^{1/2 \log|z| + i\, 1/2 \arg(z)}$$

want to ensure $\arg(z)$ well-defined on nbhd. of ∞.
rather $\arg(z^2-1)$

$$\arg(z^2-1) = \arg(z-1) + \arg(z+1)$$



choose $\arg(z^2-1) = \alpha + \beta$.

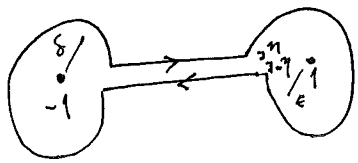
so $$\sqrt{z^2-1} = e^{1/2 \operatorname{Log}|z| + i\frac{1}{2}(\alpha+\beta)}$$

<u>Lemma</u> Along any closed curve γ in $\mathbb{C} \setminus [-1,1]$, α, β both change by <u>same</u> multiple of 2π.

$\Rightarrow$ $\frac{1}{2}(\alpha+\beta)$ changes by multiple of $2\pi i$, hence $\sqrt{z^2-1}$ unchanged.

Three <u>approaches</u>:
1. define on $|z|>1$, $\text{Im}(z) > 0$ (pos. on y-axis), $\text{Im}(z) < 0$ (neg. on y-axis) (rigorous)
2. use composition of $\sqrt{w}$ with $w = 1-z^2$ (?) $z \mapsto 1-z^2 = w$
3. geometric def'n / intuition. (??)

To compute integral use residue thm (with contour traversed clockwise)



$\gamma_{\delta,\varepsilon,\eta}$

$$\lim_{\eta\to 0} \int_{\gamma_{\delta,\varepsilon,\eta}} f(z)\,dz = \underbrace{\int_{C_1} + \int_{C_2}}_{\text{check these} \to 0 \text{ as } \delta,\varepsilon\to 0} + 2\cdot\int_{-1+\delta}^{1-\varepsilon} \frac{\sqrt{1-x^2}}{1+x^2}$$

Just have to compute residues at $\pm i$, ∞.

At $\pm i$: $\dfrac{\sqrt{2}}{2i}$, $\dfrac{-\sqrt{2}}{-2i}$

At $\underline{\infty}$: $\displaystyle -\lim_{z\to\infty} z \cdot \frac{-iz}{1+z^2}\sqrt{1-1/z^2} = i$

Residue $\underline{\text{thm}}$: $\displaystyle \int_{-1}^{1} f(x)\,dx = \left(i + \frac{\sqrt{2}}{i}\right)\cdot 2\pi i = 2\pi(\sqrt{2}-1)$. ✓

This example suggests that the right way to keep track of poles of meromorphic function is via 1-forms:

relating residues at ∞ for f to those at 0 for F

$$f(z)\,dz \quad , \quad \text{with} \quad f : \text{meromorphic}.$$

on Riemann surface — cx manifold (orientable, path-conn., 2-dim'l smooth)

Riemann sphere: compact Riemann surface.

(Any compact Riemann surface is diffeomorphic to g-holed torus, $g \geq 0$.)
g: genus.

Then Residue Thm: (notice no cycle here)

$$\sum_{z_j : \text{poles}} \operatorname{Res}_{f\,dz}(z_j) = 0.$$

(seen as precursor to Riemann-Roch thm.)

—

Other Riemann surfaces arise naturally from multi-valued functions.

e.g. $f(z) = \log z$. Try to construct Riemann surface R for which map: $R \to \mathbb{C}$, $z \mapsto \log z$ is one-one.

Draw geometric approximation.