Weierstrass product : , $\{a_n\}$ with $|a_n| \to \infty$ as $n \to \infty$,

then $\exists$ entire function with zeros at $\{a_n\}$. All such functions

are of the form:

$$f(z) = z^m \cdot e^{g(z)} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{1}{m_n}\left(\frac{z}{a_n}\right)^{m_n}}$$

for some integers m_n.
$g(z)$ entire.

<u>Note</u>: $\log\left(1 - \frac{z}{a_n}\right) = -\frac{z}{a_n} - \frac{1}{2}\left(\frac{z}{a_n}\right)^2 - \cdots$

so polynomial appearing in exponent is indeed <u>negative</u> of Taylor expansion

Examining $G(z) := \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)$ with zeros at negative integers

need to correct with $e^{-z/n}$ which suffices since

can use h terms in Taylor poly. s.t. $\sum_{n=1}^{\infty} \frac{1}{|a_n|^{h+1}}$ converges.

Recall that $\sin \pi z = \pi z \cdot \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{n}\right) e^{z/n}$

So $\sin \pi z = \pi z \; G(z) \, G(-z)$. (exploited the $n \mapsto -n$ symmetry of $\mathbb{Z}$)

Now study translation action $n \mapsto n+1$ of $\mathbb{Z}$.

$G(z-1)$ has same zeros at neg integers as $G(z)$, but also 0 at origin.

So by Weierstrass <u>Thm</u>: $G(z-1) = z \cdot G(z) \cdot e^{g(z)}$ g: entire.

To determine g, again do logarithmic diff. on both sides:

$$\sum_{n=1}^{\infty}\left(\frac{1}{z-1+n}-\frac{1}{n}\right) = \frac{1}{z} + g'(z) + \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right)$$

$n \mapsto n+1$, then re-indexing sum:

$$= \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n+1}\right) + \frac{1}{z} - 1$$

manipulation is allowable as series are abs. convergent:

$$\frac{1}{n}-\frac{1}{n+1} = \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{z+n}-\frac{1}{n}\right) + \underbrace{\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)}_{\text{sum this series, telescoping so limit of partial sums } = 1} + \frac{1}{z} - 1.$$

$\Rightarrow \quad g'(z) = 0 \quad$ i.e. $g(z)$ is constant. Call it γ.

i.e. $\gamma(z)$ constant. So may be evaluated by any choice of z in $(*)$.

$$G(0) = 1 = e^{\gamma} G(1) \Rightarrow e^{-\gamma} = \prod_{n=1}^{\infty} \left(1+\tfrac{1}{n}\right) e^{-1/n}$$

As partial products, RHS gives $\displaystyle\prod_{n=1}^{N} \underbrace{\left(1+\tfrac{1}{n}\right)}_{\frac{n+1}{n}} e^{-1/n} = (N+1)\, e^{-(1+1/2+\cdots+1/N)}$

so taking logs: $\displaystyle \gamma = \lim_{N\to\infty} \left(1+1/2+\cdots+1/N - \log N\right) \approx .57722$

(since $\lim_{N\to\infty} (\log(N+1) - \log N) = 0$) "Euler's constant"

Renormalize slightly: $H(z) := e^{\gamma \cdot z} G(z)$ so that we get the cleaner functional equation:

$$H(z-1) = z \cdot H(z)$$

$\Rightarrow \quad \Gamma(z) := 1/z H(z)$, then $\Gamma(z+1) = z\,\Gamma(z)$.

interpolates the factorial. (shifted by 1)

or as an infinite product:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}$$

Which defines meromorphic function with poles at 0, negative integers (and no zeros)

Our earlier relation $z\,G(z)\,G(-z) = \dfrac{\sin \pi z}{\pi}$ gives:

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

Set $z = 1/2$, then $\Gamma(1/2)^2 = \dfrac{\pi}{\sin \pi/2} = \pi$, i.e. $\Gamma(1/2) = \sqrt{\pi}$

these are most important relations. Also notice $\Gamma(z)\Gamma(z+1/2)$ and $\Gamma(2z)$ have same poles. Via differentiating log derivative:

$$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$

compare derivs. of log deriv. for $\Gamma(z)$ ~~[illegible]~~ v. $\Gamma(2z)$ and $\Gamma(z+1/2)$

Find these are equal, so that

$$\Gamma(z)\Gamma(z+1/2) = e^{az+b}\,\Gamma(2z)$$

do substitutions for z to find $a = -2\log 2$

$b = 1/2 \log \pi + \log 2$

$$\Rightarrow \quad \sqrt{\pi}\cdot\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\Gamma(z+1/2)$$

Legendre's duplication formula.

Possibly cleaner definition:

$$G(z) = \prod_{n=1}^{\infty} (1+z/n)\, e^{-z/n}$$

$$\Gamma(z) = \frac{1}{z}\cdot\prod_{n=1}^{\infty} (1+z/n)^{-1}\cdot e^{z/n} \cdot e^{-\gamma z}$$

γ: chosen so that $\Gamma(1)=1$.

Also Gauss' formula:

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \lim_{N\to\infty} \prod_{n=1}^{N} \frac{n\cdot e^{z/n}}{z+n}$$

$$= \lim_{N\to\infty} \frac{e^{-\gamma z}\, N!}{z(z+1)\cdots(z+N)} \exp\left(z\left(1+\cdots+\frac{1}{N}\right)\right)$$

$$= \lim_{N\to\infty} \frac{N!\,N^z}{z(z+1)\cdots(z+N)}.$$

Last time: studied $\Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n}$

obtained by inverting canonical product. Meromorphic with simple poles @ $0, -1, -2, \ldots$

Here γ = Euler's constant $= \lim_{n\to\infty} \left[\left(1 + \cdots + \frac{1}{n}\right) - \log n \right]$

Rewriting, we have $\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_n \frac{n}{z+n} \cdot e^{z/n}$

$$= \lim_{N\to\infty} \frac{e^{-\gamma z} N!}{z(z+1)\cdots(z+N)} e^{z(1+\cdots+1/N)} \qquad (*)$$

via partial products

combining $e^{-\gamma z}$ with $e^{z(1+\cdots+1/N)}$, these terms $= \underbrace{N^z}_{e^{z\log N}} \cdot e^{\underbrace{z(-\gamma + 1+\cdots+1/N - \log N)}_{\to 1 \text{ as } N\to\infty}}$

so $(*) = \lim_{N\to\infty} \frac{N!\, N^z}{z(z+1)\cdots(z+N)}$ "Gauss' Formula"

Alternate approach to Gamma function via integration.

Could have defined

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}$$

(Mellin transform of e^{-t})

$\frac{dt}{t}$ invariant meas. on $(\mathbb{R}_+, \times)$

i.e. inv. under $t \mapsto at$

for $Re(z) > 0$.

Understood as improper integral.

Want to know what properties (of convergence) are required in order to guarantee this defines analytic function, for which we may differentiate under integral sign

$f(t,z) : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$, $f(t,z_0)$ continuous for each $z_0 \in \Omega$

Then $\displaystyle\int_0^\infty f(t,z)\,dt$ is "uniformly convergent" for $z \in \Omega$

if, given ϵ, $\exists$ k s.t. $\beta > \alpha > k$,

$$\left| \int_\alpha^\beta f(t,z)\,dt \right| < \epsilon . \qquad (**)$$

<u>Thm</u>: Suppose that, for each compact $K \subseteq \Omega$, the integral $\displaystyle\int_0^\infty f(t,z)\,dt$ is uniformly conv. on K and that for each t, $z \mapsto f(t,z)$ is analytic on Ω. Then

$$F(z) := \int_0^\infty f(t,z)\,dt \quad \text{and} \quad F'(z) = \int_0^\infty \frac{d}{dz} f(t,z)\,dt$$

is analytic on Ω

<u>pf</u>: familiar to us: use Cauchy int. formula as function of z.

Get double integral in which we interchange order of integration.

See Lang. XII, §1. In particular, it ensures $F_n(z) := \displaystyle\int_0^n f(t,z)\,dt$ are unif. Cauchy seq. on Ω

Slightly <u>worse</u> for us, since $e^{-t}\, t^z \dfrac{dt}{t}$ also behaves badly at $t=0$. (not just $t=\infty$)

so as our sets $S = \{z \mid a \le \mathrm{Re}(z) \le A\}$ $0 < a < A < \infty$. $\longleftarrow$ Note these S contain all compact K in right half plane

Show estimate $(**)$ and for every $\epsilon > 0$, $\exists$ $\delta > 0$ s.t.

$0 < \alpha < \beta < \delta$, then $\left| \displaystyle\int_\alpha^\beta e^{-t}\, t^z \frac{dt}{t} \right| < \epsilon$

We'll prove latter and leave (**) to you:

for $0 < t \le 1$, $z \in \{ a \le \mathrm{Re}(z) \le A \}$

then $\left| e^{-t} t^{z-1} \right| \le t^{\mathrm{Re}(z)-1} \le t^{a-1}$

so $$\left| \int_\alpha^\beta e^{-t} t^z \frac{dt}{t} \right| \le \int_\alpha^\beta t^{a-1}\, dt = \frac{1}{a}\left(\beta^a - \alpha^a\right)$$

$\forall\ z: a \le \mathrm{Re}(z) \le A$

Choose δ so that $|\alpha - \beta| < \delta \Rightarrow \frac{1}{a}\left(\beta^a - \alpha^a\right) < \epsilon.$ //

this requires $a > 0$.

Conclusion: integral defines analytic function for $\mathrm{Re}(z) > 0$.

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To show it agrees with product formula, we show it matches

Gauss' formula for some set of real numbers with limit point.

For n integer: $x \ge 1$ real
$$\int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1}\, dt = \frac{n!\, n^x}{x(x+1)\cdots(x+n)}$$

(integration by parts)
diff. $(1 - t/n)^n$ n times
int t^{x-1} n times

So done if we can show this
integral converges to our integral repn for $\Gamma(z)$ as $n \to \infty$.

Follows because $\left(1 + \frac{w}{n}\right)^n \to e^w$ uniformly on compact sets

and $\left(1 - \frac{t}{n}\right)^n \le e^{-t}$ for all $0 \le t \le n$.

See Conway.
p. 183

$\int_{..}^n \left[\left(1 - \frac{t}{n}\right)^n - e^{-t}\right] t^{x-1}\, dt$ is small

$\int^{1/n} \left(1 - \frac{t}{n}\right)^n$ small

Even though $\Gamma(z) = \int_0^\infty e^{-t}\, t^z \, \frac{dt}{t}$ only converges for $Re(z) > 0$,

we may obtain a meromorphic continuation from the functional equation $\Gamma(z+1) = z \cdot \Gamma(z)$

which allows continuation from $Re(z) > 0$ to $Re(z) > -1$, with pole at $z = 0$,

via $\Gamma(z) := \frac{1}{z}\Gamma(z+1)$ for $Re(z) > -1$. repeating gives continuation to all of $\mathbb{C}$.

How to prove functional equation?

Integration by parts.

$$\Gamma(z+1) = \int_0^\infty e^{-t}\, t^z \, dt = \underbrace{-e^{-t} t^z \Big|_{t=0}^{t=\infty}}_{=0} + z \underbrace{\int_0^\infty e^{-t}\, t^{z-1} dt}_{\Gamma(z)}$$

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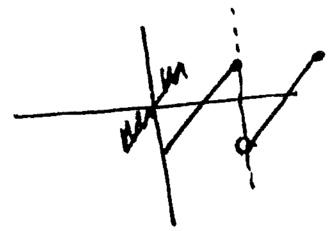
Stirling's formula: (arising from manipulations with logarithmic derivative)*

$$\log \Gamma(z) = (z - 1/2) \log z - z + \frac{1}{2} \log(2\pi) - \int_0^\infty \frac{P_1(t)}{z+t}\, dt$$

where $P_1(t) =$ "saw-tooth function" $= t - [t] - 1/2$

with $[t]$: greatest int. $\leq t$.

and log denotes principal branch.



and error term of improper integral

$\to 0$ uniformly on any sector

of cx. numbers with $-\pi + \delta \leq \arg(z) \leq \pi - \delta$, $\delta > 0$.

or as asymptotic: $\Gamma(z) \sim z^{z-1/2} e^{-z} \sqrt{2\pi}$

$\uparrow$

quotient tends to 1

as $|z| \to \infty$

(special case: $n! \sim n^n e^{-n} \sqrt{2\pi n}$ can be proven via calculus methods)

(*): $$\frac{d}{dz}\left(\frac{\Gamma'(z)}{\Gamma(z)}\right) = \sum_{n=0}^{\infty} \frac{1}{(z+n)^2}$$