8701 – Fall 2013 – Midterm Exam Review Problems

These questions are intended to be completed over the weekend and reviewed in class on Monday, October 14, in preparation for Wednesday's midterm. As further incentive, at least one of these questions will appear on the midterm itself.

1. Suppose that $f(z)$ is analytic on a region Ω, and $\mathrm{Re}(f(z))$ is constant on Ω. Must $f(z)$ must be constant on Ω?

2. In class, we noted that the function of a real variable

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is not real analytic, because its Taylor series at the origin is identically 0. Let us examine its complex analogue

$$f(z) = \begin{cases} e^{-1/z^2} & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

Show that f is analytic everywhere except $z = 0$, satisfies the Cauchy-Riemann equations at $z = 0$, but fails to be analytic at $z = 0$. Explain.

3. Give a formula for the inverse cosine function using the complex logarithm. (Hint: First express cosine in terms of the complex exponential.) Determine a principal branch of the inverse cosine function. A "branch point" may be defined as a point which must be omitted from any principal branch (so $z = 0$ is a branch point of the logarithm function). What are the branch points of the inverse cosine function?

4. Evaluate (by any means) the following integrals over the unit circle C:

$$\int_C \frac{dz}{z}, \qquad \int_C \frac{dz}{|z|}, \qquad \int_C \frac{|dz|}{z},$$

$$\int_C \frac{dz}{z^2}, \qquad \int_C \frac{dz}{|z^2|}, \qquad \int_C \frac{|dz|}{z^2}.$$

5. Evaluate the following integral over the circle $C(0,1)$ of radius 1 centered at the origin (and traversed counterclockwise):

$$\int_{C(0,1)} \bar{z}\, dz$$

6. Suppose that γ is a simple closed curve, and that $f(z)$ is an analytic function in a region Ω containing γ. Assume $f'(z)$ is continuous on Ω. Show that

$$\int_\gamma \overline{f(z)} f'(z)\, dz$$

is purely imaginary.

7. Suppose that $f(z)$ is analytic for $|z| < 1$ and $|f(z)| \leq \frac{1}{1-|z|}$. What is the best estimate for $|f^{(n)}(0)|$ using Cauchy's inequality that is obtainable from this information?