Math 8701 – Fall 2013 – Problem Set 7

1. Minimum modulus principle – If f is a non-constant analytic function on a bounded open set Ω, continuous on its closure, then either f has a zero in Ω or $|f|$ assumes its minimum value on the boundary of Ω.

2. Let f be an analytic function on a bounded, open, connected set Ω, and continuous on its closure. Show that if there exists a constant $c \geq 0$ such that $|f(z)| = c$ for all z on the boundary of Ω, then either f is a constant function or f has a zero in Ω.

3. Let f be an analytic function on the half-strip defined by

$$\{z \mid \mathrm{Im}(z) \in [-\pi/2, \pi/2], \mathrm{Re}(z) \geq 0\}.$$

Suppose that

$$|f(z)| \ll e^{e^{C\mathrm{Re}(z)}}, \qquad \text{for some constant } C \text{ with } 0 \leq C < 1,$$

and that $|f(z)| \leq 1$ on the boundary of the half-strip. Show that $|f(z)| \leq 1$ for all points z in the half-strip.

4. Let Ω be an open, connected set and let $\gamma_a(t) \equiv a$ denote the constant curve that is identically equal to $a \in \Omega$ for $t \in [0,1]$. Show that if a (smooth) closed curve γ is homotopic to γ_a, then γ is homotopic to the constant curve $\gamma_b \equiv b$ for any other point $b \in \Omega$. (Thus, when we say that a closed curve is "homotopically trivial" we need not specify a point in Ω to which it deforms.)

5. Show that if we change the definition of *homotopic* given in class, by removing the restriction that $\Gamma(0,t) = \Gamma(1,t)$ for all $t \in [0,1]$, then we can find two curves which are "homotopic" (in this altered sense) in $\mathbb{C} - \{0\}$, but have different line integrals for some function f on $\mathbb{C} - \{0\}$. (Thus, the general form of Cauchy's theorem would be false, as stated, with this modified definition of homotopy.)