Deformations of characters, metaplectic Whittaker functions, and the Yang-Baxter equation

by

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Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2010

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Abstract

Recent work has uncovered an unexpected connection between characters of representations and lattice models in statistical mechanics. The bridge was first formed from Kuperberg's solution to the alternating sign matrix (ASM) conjecture. This conjecture enumerates ASMs, which can be used to describe highest weight representations, but Kuperberg utilized a square ice model from statistical mechanics in his proof. Since that work, other results using similar methods have been demonstrated, and this work continues in that vein.

We begin by defining the particular lattice model we study. We then imbue the lattice model with Boltzmann weights suggested by a bijection with a set of symmetric ASMs. These weights define a partition function, whose properties are studied by combinatorial and symmetric function methods over the next few chapters. This course of study culminates in the use of the Yang-Baxter equation for our ice model to prove that the partition function factors into a deformation of the Weyl denominator in type B and a generalized character of a highest weight representation. Finally, the last two chapters deal with two approaches to computing Whittaker coefficients of Eisenstein series and automorphic forms.

Thesis Supervisor: Benjamin B. Brubaker
Title: Cecil and Ida B. Green Career Development
Associate Professor of Mathematics

Acknowledgements

Acknowledgements here.

Contents

Chapter 1

Introduction

This thesis studies Whittaker coefficients of metaplectic forms, using three very different methods each associated to a different classical Lie group. In the main component of the thesis we investigate a deformation of highest weight characters of $\mathrm{SO}(2r+1)$ using methods of statistical mechanics, similar to the recent work of Brubaker, Bump, and Friedberg [5] for Cartan type A. In addition, we explore Whittaker coefficients of metaplectic Eisenstein series on $\mathrm{Sp}(2r)$ using an explicit factorization of elements in the unipotent radical of a certain maximal parabolic subgroup. For this, we draw inspiration from the paper of Brubaker, Bump, and Friedberg [4], which again focuses on the type A case. Finally, we make a brief exploration of Hecke operators on the metaplectic group in a specific example – the six-fold cover of $\mathrm{GL}(3)$ – and draw a few conclusions about the orbits of Whittaker coefficients under two operators which generate the Hecke algebra. We now describe these results, and the previous work that motivated them, in more detail.

In 1996, Kuperberg [13] produced a remarkable proof of the deceptively simple alternating sign matrix (ASM) conjecture. The conjecture of Mills, Robbins, and Rumsey [16], first proven by Zeilberger [20], enumerates sets of ASMs, matrices with entries $0, 1$, or -1 satisfying certain conditions. The significance of the proof was its use of techniques from statistical mechanics. In particular, Kuperberg employed a combinatorial equivalence between ASMs and states of a two dimensional square lattice model known as the six-vertex model or square ice. Following ideas of Baxter [2], he used the Yang-Baxter equation of this model, proven by Izergin and Korepin [**?**], to evaluate a partition function relevant to the conjecture. This thesis exhibits a similar connection between statistical mechanics and the combinatorics of representation theory in Cartan type B.

Our appoach is motivated by the work of Tokuyama [19], later expanded by Okada

[17], which concerns the formulation of generating functions identities for characters and Weyl denominators. Tokuyama discovered the following identity, a deformation of the Weyl character formula in Cartan type A.

$$\sum_{T \in SG(\lambda)} (t+1)^{s(T)} t^{l(T)} \prod_{i=1}^{r} (z_i^{m_i(T)}) = \left[\prod_{1 \leq i < j \leq r} (z_i + z_j t) \right] \cdot S_{(\lambda-\rho)}(z_1, \ldots, z_r) \tag{1.1}$$

The generating function on the left side of (1.1) is summed over strict Gelfand-Tsetlin patterns (GTPs), triangular arrays of integers whose rows decrease and interleave (see Definition 2.5), with fixed top row the strict partition λ. The functions s, l and m_i are statistics on the entries of GTPs. On the right side, $S_{\lambda-\rho}$ is the Schur function associated with the partition $\lambda - \rho$ for $\rho = (r, r-1, \ldots, 2, 1)$. The partition $\lambda - \rho$ may be viewed as a highest weight of $\mathrm{GL}_r(\mathbb{C})$. So the RHS is the product of a deformation of the Weyl denominator in type A and a character of a representation of $\mathrm{GL}(\mathbb{C})$ with highest weight corresponding to the set of Gelfand-Tsetlin patterns.

Brubaker, Bump, and Friedberg developed a square ice interpretation of this formula [5]. They were able to show that the partition function of their ice model satisfied a similar identity to Tokuyama's generating function by proving a Yang-Baxter equation for their model, drawing from [2].

Okada established similar generating functions to Tokuyama's in Cartan types B, C, and D, using ASMs with certain symmetry properties as indices, to produce deformations of the Weyl denominator formulas in these cases [17]. In this thesis, we start with the work of Okada in Cartan type B, but paralleling Brubaker, Bump, and Friedberg, trade the index of ASMs with rotational symmetry for a model of square ice we develop, called half-turn ice. By showing a bijection and choosing Boltzmann weights related to the statistics of Okada, we turn the generating function into a partition function. Then, in the style of Baxter [2], we determine certain star-triangle relations on the ice which help factor the partition function into a deformed Weyl denominator and a polynomial, which we show to be a deformed character. This relationship is stated as follows.

Theorem 3.10. *We may write the partition function Z_λ of half-turn ice indexed by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ as*

$$Z_\lambda(x_i; t_i) = \left[\prod_{i=1}^{r} (x_i - t_i) \right] \cdot \left[\prod_{1 \leq j < k \leq r} (x_k - x_j t_j t_k)(x_j x_k - t_j t_k) \right] \cdot \phi_\lambda(x_i; t_i), \tag{1.2}$$

for ϕ_λ a Weyl-invariant polynomial.

Then Theorem 5.15 shows that the special value $\phi_\lambda(x_1, \dots, x_r; 1, \dots, 1)$ is a character of $SO_{2r+1}(\mathbb{C})$.

We also prove that these deformed characters ϕ_λ satisfy a certain formulation of Clebsch-Gordan decomposition, using several original proof techniques that appeal to the combinatorics of square ice. With some small computable examples we have stated a conjectural deformation of Pieri's Rule for them as well, and along with a conjectured "base case," a recursive formula for these deformed characters for all λ would follow. The precise meaning of these deformed characters is still unknown. In the type A case [5], the extra factor is a Schur polynomial, which can be considered the value of a Whittaker coefficient by the Casselman-Shalika formula. If our case were to parallel this, we would expect a Whittaker coefficient of some metaplectic cover of $\mathrm{SO}(2r+1)$ or $\mathrm{Sp}(2r)$, but these coefficients have yet to be computed.

The content of the final two chapters of the thesis shifts focus significantly. We describe two methods of computing Whittaker coefficients of automorphic forms. The first studies the reduction of the Whittaker integrals of rank two Eisenstein series on Sp_4, and more generally Sp_{2r}, to lower rank by a recursion. This approach is inspired from the type A case studied in [4], but the details of the computation differ from that case significantly. Our result reduces the Sp_4 Whittaker coefficients to combinations of SL_2 coefficients and certain exponential sums H_ψ:

Theorem 6.9. $a_f^{\mathrm{Sp}(4)}(m_1, m_2; \psi) = \sum_{d_1,d_2,d_3 \in \mathfrak{O}_S \setminus \{0\}/\mathfrak{O}_S^\times} H_\psi(d_i) a^{\mathrm{SL}(2)}_{f^{(m_1)}_{d_1,d_3,d_2}} \left(\frac{d_3^2}{d_2^2} m_2; \psi \right)$.

The second computational method, exemplified over the six-fold cover of GL_3 in the last chapter, is joint work with Cathy Lennon. Our strategy takes advantage of a metaplectic form being an eigenfunction of Hecke operators. We explicitly compute the Whittaker integral of a form acted on by each of two generators of the metaplectic Hecke algebra. The first of these generators was computed by Hoffstein in [9], where it was shown that the Whittaker coefficients have four independent orbits of dependencies. We determine that the second generator gives the same orbits, by finding a bijective map between the sets of right coset representatives of the double cosets of the two generators. The explanation of this redundancy of information is observed to be the equivalence of choosing the two orderings of the positive simple roots of SL_3. The work over Sp_4 by Brubaker and Friedberg in [**?**] was a helpful reference for this calculation.

Chapter 2

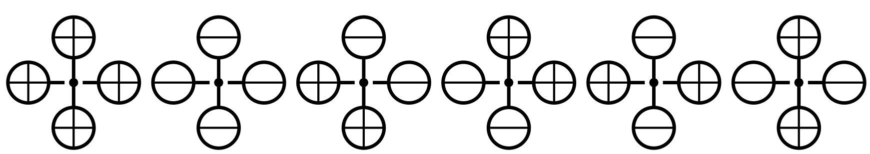
Combinatorial background

Our purpose in this chapter is to define half-turn ice, a combinatorial structure to be used in a two-dimensional model (in the sense of statistical mechanics). As we will explain, this model will be closely connected to the representations of complex Lie groups of Cartan type B. Our construction generalizes that of Kuperberg [14] and is a natural extension of that for type A in [5]. To motivate the construction, we begin by describing three combinatorial objects: square ice, alternating sign matrices, and Gelfand-Tsetlin patterns. We then show certain bijective equivalences between subsets of these three objects, and these connections will guide the invention of half-turn ice.

2.1 Square ice

The square ice in this paper is a variation of the six-vertex model detailed in [2]. The particular notations and diagrams we use emulate [5].

An arrangement of square ice is a finite square grid whose edges are assigned a positive or negative sign. In general, any given vertex has $2^4 = 16$ possible arrangements of signs on the four adjacent edges, but only six of these combinations, called *vertex fillings*, are allowed. These are shown in (2.1).

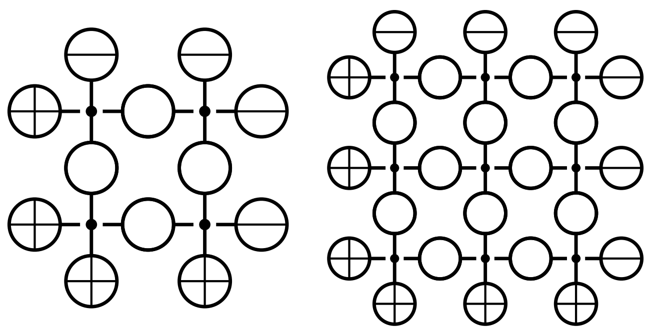


(2.1)

Remark 2.1. There are equivalent formulations which give the edges a direction [14]. In those models, the six vertex fillings in (2.1) are those having two incoming edges and two outgoing edges. The choice of $+$ and $-$ in our formulation amounts to a choice of orientation of arrows.

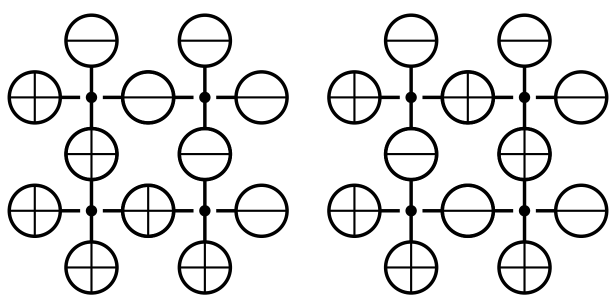
We consider finite pieces of ice within a certain boundary. Square ice boundaries are indexed by distinct partitions λ. A *partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a finite nonincreasing sequence of positive integers. A partition is *distinct* if it is strictly decreasing (i.e. none of the integers repeat).

First, fix the indexing partition as $(r, r-1, r-2, \ldots, 2, 1)$ for some fixed integer r. Throughout the paper partitions of this form for a given r will be called ρ, so this is the $\lambda = \rho$ case. Then *ρ-square ice* is a square piece of r^2 vertices in r rows and r columns. The boundary edges are those that only have one of their two vertices in the $r \times r$ array; to these we assign with the following signs. The bottom and left boundary edges have a $+$ sign, and the top and right boundary edges have a $-$ sign. The following examples show the type A ρ-square ice boundaries for $r = 2$ and $r = 3$.



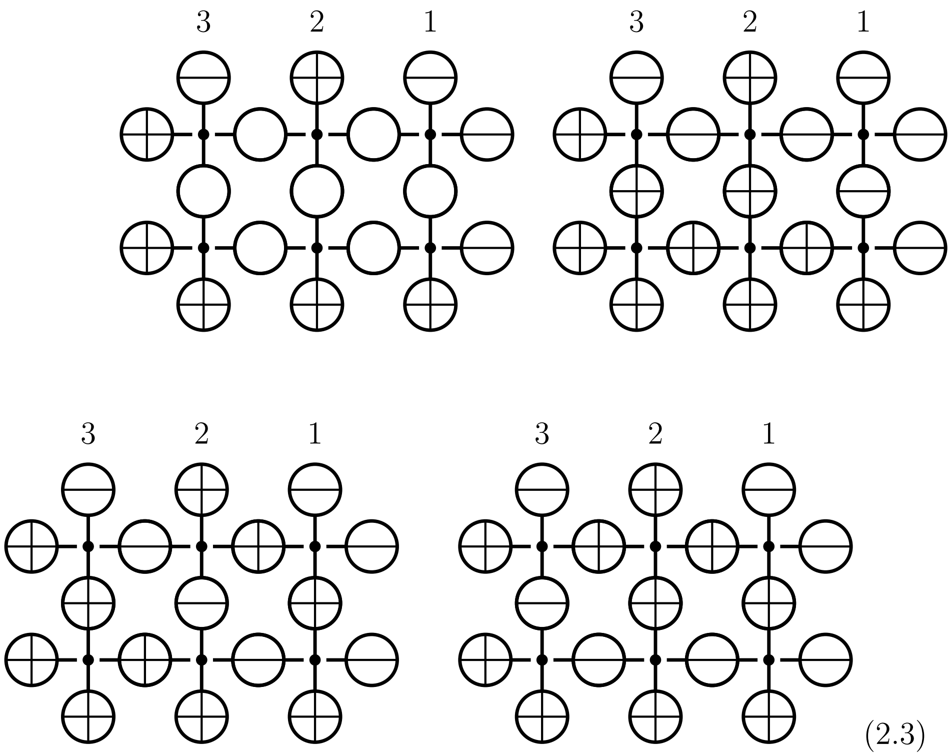
(2.2)

Once we have given boundary conditions for a piece of square ice, we can consider its collection of *fillings*, which are assignments of $+$ or $-$ signs to the interior edges such that all r^2 vertices have one of the six allowed vertex fillings. For example, the $r = 2$ ρ-square ice has the following two fillings.



Now, for a general distinct partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we alter the dimensions and boundary signs of our finite grid of square ice in the following way. The piece of ice is now rectangular, with r rows and λ_1 columns of vertices (remember that in this paper partitions are always written in decreasing order, so λ_1 is the largest part of λ). We assign a $+$ to the boundary edges on the left and bottom sides, and a $-$ on the right side. This leaves the top boundary to be assigned.

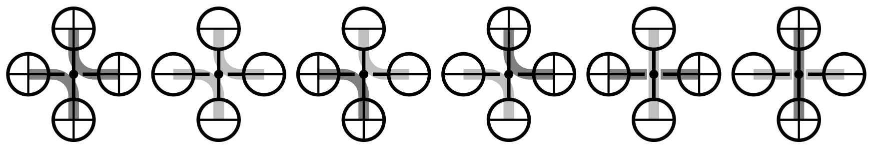
We label the columns of vertices $1, 2, 3, \ldots, \lambda_1 - 1, \lambda_1$ in ascending order from **right to left**. Each top boundary edge in a column whose label is some λ_i is assigned a $-$ sign, all others are assigned a $+$ sign. As an example, here is the boundary for the $(3, 1)$-square ice, and its three fillings.



(2.3)

A distinct partition λ contains all of the information about the boundary of a piece of square ice. The number of terms, or *length*, of λ gives the number of rows r of the piece of ice, and the largest term, λ_1, gives the number of columns. The columns are labeled from the right, and we know all boundary values by knowing the entries of λ.

Remark 2.2 (Flowlines). Here we observe a useful property of square ice fillings. In (2.1), we see that the allowed vertices are exactly the ones that have the same total number of $+$'s on their top and left edges as they do on their bottom and right edges. This conservation can be illustrated globally in any given filling by overlaying *flowlines* at each vertex which depend on its vertex filling.



(2.4)

These lines connect like signs on adjacent edges. Once they are connected in a filling, the signs can be thought of as ‘flowing’ from the top left of the filling to the bottom right, following the flowlines. We note these three principles of flowlines.

1. The sign of a flowline is constant, so it is determined by the boundary edge through which the flowline enters the piece of ice.

2. Flowlines always flow down or to the right, and each edge is covered by exactly one flowline.

3. Flowlines of like signs never cross, but flowlines of unlike signs may or may not cross.

The restriction of the square ice fillings to be composed of the six vertex fillings (2.1) is equivalent to the flowlines of sign being restricted to the forms in (2.4). Thus the three principles of flowlines encode the allowable fillings of a piece of square ice.

2.2 Alternating sign matrices

Definition 2.3. An $r \times r$ matrix A is an *alternating sign matrix* (ASM) if the following three conditions hold.

1. Every entry of A is $0, 1$, or -1.

2. In each row and column, the nonzero entries of A alternate between 1 and -1.

3. Every row and every column of A has entries that sum to 1.

These three conditions imply that in each row and column, the first and last nonzero term is 1. Also, we have that any partial sum of a row or column will be either 0 or 1. Some examples of matrices of this type are given below. Note that any permutation matrix is an alternating sum matrix.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Mills, Robbins, and Ramsey [16] conjectured the following beautiful formula for the number of $r \times r$ ASMs.

$$\#\{r \times r \text{ ASMs}\} = \frac{1! \cdot 4! \cdot \cdots \cdot (3r-5)! \cdot (3r-2)!}{r! \cdot (r+1)! \cdot \cdots \cdot (2r-2)!(2r-1)!}$$

A proof of this formula was only found fairly recently. The highly computational first proof was due to Zeilberger [20]. Later, Kuperberg [13] found a much shorter proof utilizing a six-vertex model equivalent to the ρ-square ice presented earlier.

We generalize to nonsquare ASMs.

Definition 2.4. For a distinct partition λ with first (largest) entry n and length r, a λ-*alternating sign matrix* A is an $r \times n$ matrix satisfying the following conditions.

1. Every entry of A is $0, 1$, or -1.
2. In each row and column of A, the nonzero entries begin with 1 and alternate between 1 and -1.
3. Every row of A and the columns of A that, when counted from the **right** are entries in λ, have sum 1, and the other columns have sum 0.

Note that from Condition 2, the columns that sum to 0 still must have first (topmost) nonzero entry equal to 1. As an example, the following three matrices are a complete list of the $(3, 1)$-ASMs:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

2.3 Gelfand-Tsetlin patterns

The third combinatorial objects of interest are Gelfand-Tsetlin patterns. Gelfand-Tsetlin patterns were originally used to describe characters for highest weight representations of $\mathrm{GL}_{r+1}(\mathbb{C})$ by making use of the multiplicity-free branching rule from GL_{i+1} to GL_i [8]. These patterns were generalized to classical groups by Zhelobenko [21] (see also Proctor [18]), and are often still called Gelfand-Tsetlin patterns.

Definition 2.5. A *type A Gelfand-Tsetlin pattern* (GTP) is a triangular array of positive integers of the form

$$\left\{\begin{array}{ccccccccc} a_{1,1} & & a_{1,2} & & a_{1,3} & & \cdots & & a_{1,r} \\ & a_{2,1} & & a_{2,2} & & \cdots & & a_{2,r-1} & \\ & & a_{3,1} & & \cdots & & a_{3,r-2} & & \\ & & & \ddots & & \iddots & & & \\ & & & & a_{r,1} & & & & \end{array}\right\}$$

with weakly decreasing rows and $a_{i,j}$ satisfying the interleaving condition:

$$a_{i,j} \geq a_{i+1,j} \geq a_{i,j+1} \qquad \forall 1 \leq i,j \leq r-1 \text{ and } i+j \leq r. \tag{2.5}$$

If the entries $a_{i,j}$ of the pattern are strictly decreasing in rows, we say the Gelfand-Tsetlin pattern is *strict*. They are referred to as *monotone triangles* in [17]. We will deal almost exclusively with strict GTPs.

We index strict GTPs by their top row, a distinct partition. So we may consider the finite collection of strict λ-GTPs, those that are indexed by λ.

2.4 Bijections

Now we exhibit bijections between these combinatorial objects. One of the uses of strict Gelfand-Tsetlin patterns in this paper is to establish the bijective link between square ice fillings and ASMs detailed below.

For a distinct partition λ, we define a function Φ_λ from strict λ-GTPs to λ-square ice fillings. Let Φ_λ map the strict Gelfand-Tsetlin pattern $\{a_{i,j}\}$ to the filling whose signs lying on columns between the $i-1$st and ith rows are $-$'s in exactly the columns labeled with the numbers $a_{i,j}$ for $j = 1, 2, \ldots, r+1-i$. For example, the $(3,1)$-square ice fillings in (2.3) are the respective $\Phi_{(3,1)}$-images of the strict $(3,1)$-GTPs below.

$$\{\text{unfilled pattern}\} \qquad \left\{\begin{array}{ccc} 3 & & 1 \\ & 1 & \end{array}\right\}$$

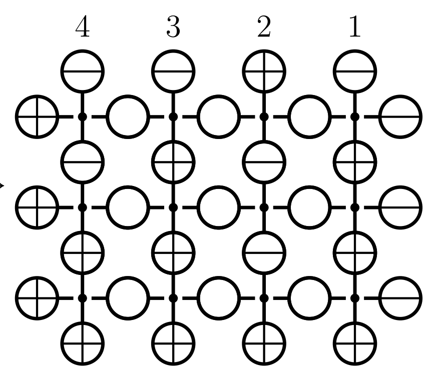
$$\left\{\begin{array}{ccc} 3 & & 1 \\ & 2 & \end{array}\right\} \qquad \left\{\begin{array}{ccc} 3 & & 1 \\ & 3 & \end{array}\right\}$$

Lemma 2.6. *For λ any distinct partition, Φ_λ is a bijection between strict λ-Gelfand-Tsetlin patterns and λ-square ice fillings.*

Proof. This is a result of Lemma 1 of [5], but we give a slight variant of the proof here.

Given a GTP $(a_{i,j})$ with top row λ, to find its image under Φ_λ, we first fill in the vertical edges of a λ-square ice piece according to the $a_{i,j}$. For example,

$$\Phi_{(4,3,1)} : \left\{\begin{array}{ccccc} 4 & & 3 & & 1 \\ & 4 & & 2 & \\ & & 2 & & \end{array}\right\} \longmapsto$$



Once the vertical edges are filled in, the signs of the horizontal edges in the filling are determined by the restriction that we use only the six allowed vertex fillings. To see this, we may employ the flowline description of square ice from Remark 2.2. In the ith row of horizontal edges, the negative flow entering in the column labeled $a_{i,j}$ must flow out the column labeled $a_{i+1,j}$, except for the rightmost flow. This last flow enters in the column labeled $a_{i,r+1-i}$ and flows out the rightmost horizontal edge.

The principle that flowlines always flow down and right is equivalent to the first of the interleaving inequalities (2.5). The fact that like-signed flowlines never cross one another is equivalent to the second. ■

Finally, Okada ([17], Proposition 1.1) gives a correspondence between λ-ASMs and strict λ-GTPs, which we will not make explicit use of here.

Lemma 2.7. *For any distinct partition λ, there is a canonical bijection between the set of λ-ASMs and the set of strict Gelfand-Tsetlin patterns with top row λ.*

With a bijective correspondence among the three combinatorial objects, one might ask what is the use of three ‘equivalent’ ideas. Throughout the paper, we will demonstrate that the different mathematical frameworks allow for different approaches, arguments, and generalizations, in such a way that they all are of help. Here we will list a few key differences between the three objects, and the strengths of each.

- **Square Ice** is the most locally defined of the three objects. That is, checking the allowability that any given filling of a fixed boundary involves only looking at the four edges out of each vertex independently of the other vertices. This allows for ‘local’ arguments to be very small, for example the star-triangle identity below that only involves three vertices. The locality of square ice also makes it easier to generalize, because it more flexibly fits into an arbitrary boundary.

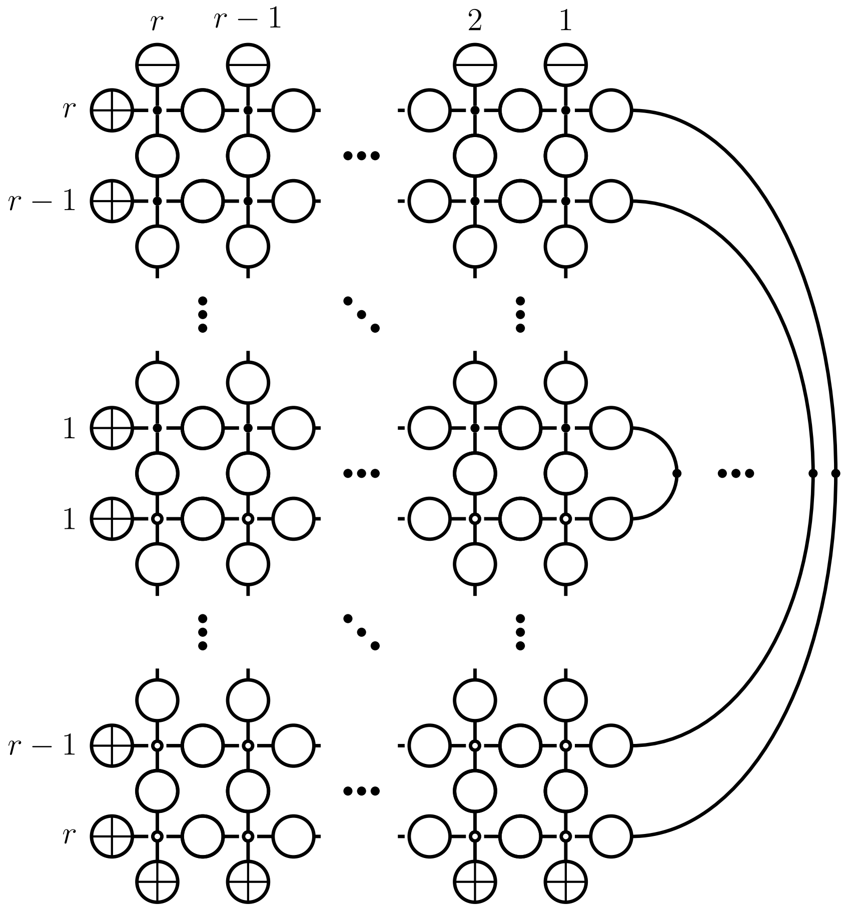
- In contrast, **Alternating Sign Matrices** are the least locally defined of the three objects. This is because the information about the alternation of the signs requires looking at arbitrarily many entries of the matrix simultaneously, either horizontally or vertically. However, we take advantage of the connections between ASMs and representation theory from the work of Okada [17] and Tokuyama [19].

- Finally, **Gelfand-Tsetlin Patterns** have already been useful by helping draw the bijection between the other two objects. In general, GTPs lend themselves to arguments by induction on rank, according to the branching rule. We return to this later.

2.5 Type B versions

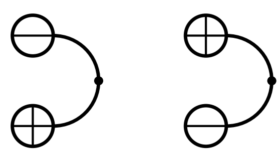
We now define the versions of square ice and ASMs for Cartan type B, which we refer to as half-turn ice and half-turn symmetric ASMs, respectively. We draw inspiration from [17] and [18] for the proper formulations in the ρ case.

Definition 2.8 (Half-turn ice)**.** The type B version of ice is λ-*half-turn ice*. The lattice structure of square ice is still used, but the shape and boundary conditions of the piece of ice are changed. For $\lambda = (r, r-1, \ldots, 2, 1)$ has the following shape and boundary values.



(2.6)

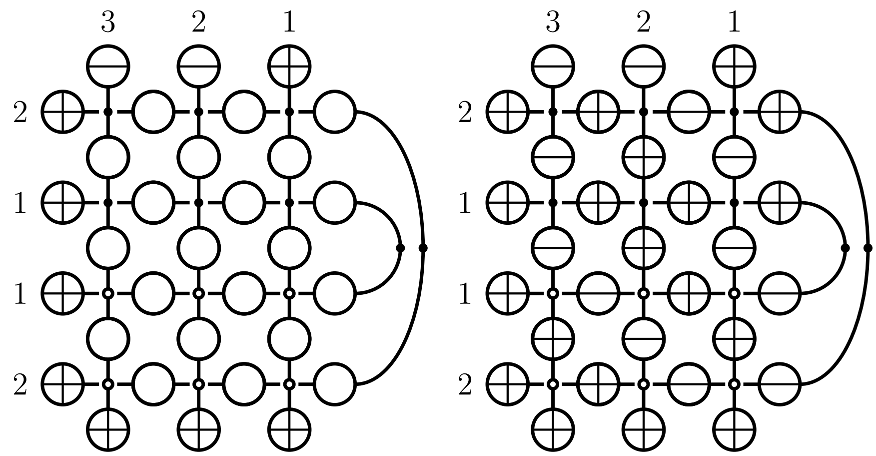
Notice that the number of rows is now $2r$, twice the length of the partition. Also, the right boundary has U-shaped vertices with two edges linking the ith and $(2r+1-i)$th rows. Fillings of half-turn ice must use the same six vertex fillings (2.1) of square ice throughout, with the following extra condition concerning the right boundary. A filling requires each of these U-shaped vertices to have one of the following two vertex fillings.



(2.7)

This implies that the right boundaries of half-turn ice have 2^r allowable variations. Finally, in Figure (2.6) there are two colors for the vertices. The meaning of these shadings will be discussed later.

To generalize this ice to arbitrary distinct partitions λ, we insert columns just as we did in the square ice. So the λ-half-turn ice boundary has $2r$ rows and λ_1 columns, for λ_1 the largest part of λ. The columns are indexed from the right exactly as in the type A case, so the column of vertices attached to the half-turn pieces are in column 1, and the leftmost column is labeled λ_1. The top border again has $-$'s exactly in each column that is labeled with a part of λ and $+$'s elsewhere. The left and bottom boundaries are assigned all $+$'s, and the right boundary has exactly the same restriction: each filling must assign values to the U-shaped vertices so they are of one of the forms in Figure (2.7). Here is an example of the $\lambda = (3, 2)$ boundary and one possible filling (of 35 total fillings).



Definition 2.9 (Half-turn symmetric ASMs)**.** Now we consider the type B version of alternating sign matrices, which are detailed in Kuperberg's work [14] for the $\lambda = \rho$ case. Kuperberg calls them $2r \times 2r$ HTSASMs, which stands for *half-turn symmetric alternating sign matrices.* As the name implies, a $2r \times 2r$ alternating sign matrix $A = (a_{i,j})$ is a HTSASM if $a_{i,j} = a_{2r+1-i,2r+1-j}$ for all i, j.

Lemma 2.10. *For any r, $2r \times 2r$ HTSASMs are in bijective correspondence with ρ-half-turn ice fillings.*

Proof. This is implicit in Kuperberg [14], but we give a brief constructive argument. Given a $(r, r-1, \ldots, 2, 1)$-half-turn ice filling, we can extend it to a $(2r, 2r-1, \ldots, 2, 1)$-square ice filling by taking two copies of the half-turn ice filling, reversing all the signs in one copy, and then attaching the reversed copy to the original along the half-turn boundary (first rotate the reversed copy a half-turn). The restriction on the fillings of the U-shaped vertices ensures that the signs match up, and so the curved edges straighten out and we are left with a $2r \times 2r$ square ice filling invariant under rotation by a half-turn and sign reversing.

Chapter 3

Boltzmann weights and the partition function in type B

We are studying these combinatorial objects due to their connection to representation theory. As noted above, in type A, Gelfand-Tsetlin patterns parametrize basis vectors of highest weight representations of $\mathrm{GL}_r(\mathbb{C})$. The highest weight may be read off the top row, regarded as an element in the weight lattice with the usual identification with $\mathbb{Z}^r$. Each pattern having this top row is a basis vector in the given highest weight representation, and patterns having identical row sums are in the same weight space (see [18]). However, the representation theoretic meaning of square ice is more subtle, since these are in bijection with *strict* GTPs.

In type A, a much deeper connection between square ice and representation theory was given by Hamel and King [10], building on work of Tokuyama [19]. They showed that the partition function for square ice produces the character of a highest weight representation, up to a deformation of the Weyl denominator formula, in Cartan types A and C. This was later studied using techniques of statistical mechanics, particularly the Yang-Baxter equation, by Brubaker, Bump, and Friedberg in type A [5] and Ivanov in type C [11].

We will demonstrate a similar result leading to a deformation of a highest weight character in type B using the half-turn ice defined above. We begin with the general definition of Boltzmann weights and the partition function on ice.

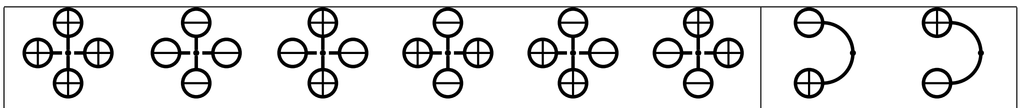
Definition 3.1 (Boltzmann weights)**.** In an ice model with a set of allowed vertex fillings U, we define a *set (or choice) of Boltzmann weights* for the ice model to be the assignment to each element $u \in U$ a rational function $k_u(x,t) \in \mathbb{C}(x,t)$. Given a piece of ice with rows indexed by $\{1,\ldots,r\}$ and a vertex v in row i, each filling T

of the piece of ice assigns to v some vertex filling u. Then for a set of Boltzmann weights k_u, the *Boltzmann weight of* v in the filling T is

$$B(v, T) = k_u(x_i, t_i).$$

Finally, the *evaluation* of a filling T of a piece of ice is defined to be the product of the Boltzmann weights of its vertices. In the literature, this may be called the Boltzmann weight of a filling.

So, applying this definition to our half-turn ice model, we need to choose polynomials for the six vertex fillings in (2.1), and the two U-shaped vertices in (2.7). We label the Boltzmann weights as follows.



$a_1(x,t)$	$a_2(x,t)$	$b_1(x,t)$	$b_2(x,t)$	$c_1(x,t)$	$c_2(x,t)$	$d_1(x,t)$	$d_2(x,t)$

(3.1)

Now, given an ice model with a set of Boltzmann weights, we can define the partition function of a piece of ice with boundary conditions.

Definition 3.2 (The partition function)**.** Given a piece of ice with fixed boundary conditions and rows indexed from 1 to r, we define its *partition function* $Z(\boldsymbol{x}; \boldsymbol{t})$ for $\boldsymbol{x} = (x_1, \ldots, x_r)$ and $\boldsymbol{t} = (t_1, \ldots, t_r)$ as

$$Z(\boldsymbol{x}; \boldsymbol{t}) = \sum_T \prod_v B(v, T). \tag{3.2}$$

In (3.2), the sum is over all allowable fillings T of the piece of ice and the product is over v vertices of the piece of ice. The product is the evaluation of the filling T.

Remark 3.3. A note about the word *partition*, which has two different meanings in this paper. The first meaning is the number theoretic definition of a partition: an expression of a positive integer as the unordered sum of positive integers. In this paper we notate this as a non-increasing tuple of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. The second meaning is the statistical mechanical definition of a partition function: a quantity encoding the statistical properties of a model. We denote this as $Z = Z(\boldsymbol{x}; \boldsymbol{t})$, and always use the full name “partition function.” The word is well established in both fields of study, so we decided to use it in both cases. We hope this does not cause confusion.

3.1 Okada's theorem and a choice of Boltzmann weights

To choose a set of Boltzmann weights in type B, we draw inspiration from the work of Okada. In [17], he develops statistics on certain sets of HTSASMs which allow him to write deformations of the Weyl denominator in types B and C as sums indexed by these sets. These statistics on HTSASMs, once translated to functions on ρ-half-turn ice fillings, will serve as our Boltzmann weights. First we state the relevant theorem of Okada (i.e., the one concerning type B).

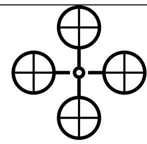
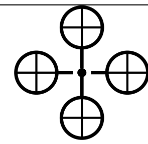
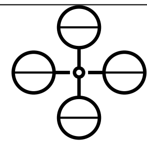
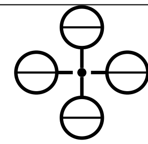
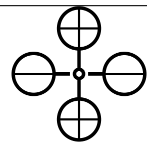
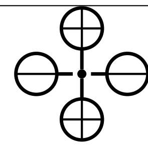
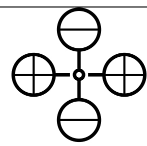
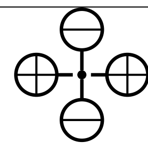
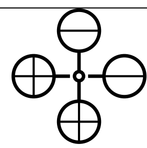
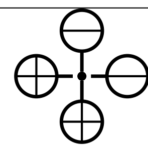
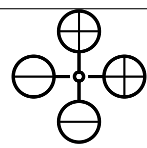
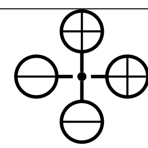
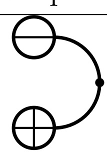
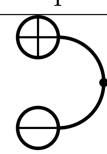
Let $\mathcal{B}_r$ denote the set of $2r \times 2r$ HTSASMs. Okada defines a set of four statistics $i_1^+, i_2, i,$ and s on $2r \times 2r$ HTSASMs, certain functions from $\mathcal{B}_r \mapsto \mathbb{R}$ detailed in [17], we have the following result.

Theorem 3.4 ([17], Theorem 2.1)**.**

$$\prod_{i=1}^{r}(1 - tx_i) \prod_{1\le i<j\le r}(1 - t^2 x_i x_j)(1 - t^2 x_i x_j^{-1}) \\ = \sum_{A\in\mathcal{B}_r} (-1)^{i_1^+(A)+i_2(A)/2} t^{i(A)} \left(1 - \frac{1}{t^2}\right)^{s(A)/2} x^{\delta(B_r)-A\delta(B_r)}, \qquad (3.3)$$

where $\delta(B_r) = {}^t(\, r - \frac{1}{2}, r - \frac{3}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, \ldots, -(r - \frac{1}{2}))$ *and* $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_r^{\alpha_r}$ *for* $\alpha = {}^t(\, \alpha_1, \ldots, \alpha_r, -\alpha_r, \ldots, -\alpha_1)$*.*

We state this result to give the precise deformation of the Weyl denominator (LHS of (3.3)) appearing in [17] and to give the general shape of the generating function. To avoid a long digression, we will omit the definition of the particular statistics $i_1^+, i_2, i,$ and s. The important point is that they may be translated from statistics on entries of HTSASMs to Boltzmann weights on square ice in such a way that the sum on the RHS of (3.3) can be written as a partition function on ρ-half-turn ice. Using the bijective correspondence established in Lemma 2.10, this can be done. The resulting Boltzmann weights are in the following table.

Clear ice	Weight	Black ice	Weight
	x_i		1
	1		x_i
	$t_i\sqrt{-1}$		$x_i t_i\sqrt{-1}$
	$x_i t_i\sqrt{-1}$		$t_i\sqrt{-1}$
	1		1
	$x_i(1-t_i^2)$		$x_i(1-t_i^2)$
U-shaped ice	**Weight**	**U-shaped ice**	**Weight**
	1		$\sqrt{-1}$

(3.4)

We note a few important details about the Boltzmann weights in the table.

- The normal four-edged vertices used in square ice come in two versions: *clear ice* (drawn as a hollow dot) and *black ice* (drawn as a solid dot). These two versions are used to differentiate the weights used on vertices in the top half of the half-turn ice with those used on vertices in the bottom half appearing in (2.6). This is necessary because the statistics on the HTSASMs from [17] we need to encode differentiate between the two independent pairs of quadrants of the HTSASM.
- The complex number $\sqrt{-1}$ appears in some of the weights.

- The variables x and t in the weights are subscripted by i, the index of the vertex. As explained in Definition 3.1, this index corresponds to the row in which the vertex belongs. The top half of the $2r$ rows are labeled r down to 1 from top to bottom. Then, the rows bend around by the half-turn pieces, so the bottom half of the rows are labeled 1 back up to r from top to bottom. The indices are labeled in (2.6).

- The U-shaped vertices on the right boundary of the half-turn ice have constant weights; they are independent of i, the index of the row.

These Boltzmann weights were selected so the partition function of ρ-half-turn ice would match the deformed Weyl denominator formula, by Theorem 3.4. The advantage to the half-turn ice formulation is that its Boltzmann weights, being completely local data, can equally apply to λ-half-turn ice for general distinct partitions λ. We study this particular partition function now.

3.2 The half-turn ice partition function

Given any distinct partition λ, we assign boundary conditions for half-turn ice as above and use Boltzmann weights as in (3.4). Then let $Z_\lambda(\boldsymbol{x};\boldsymbol{t})$ denote the resulting partition function as defined in (3.2). Z_λ is a (finite-degree) polynomial in the $2r$ variables $\boldsymbol{x} = (x_1, \ldots, x_r)$ and $\boldsymbol{t} = (t_1, \ldots, t_r)$.

To compare with the result of Okada, we set $t_i = t$ for $1 \leq i \leq r$, and set $\lambda = \rho$.

$$Z_\rho(\boldsymbol{x}; t) = \prod_{i=1}^{r}(x_i - t) \prod_{1 \leq i < j \leq r} (t^2 - x_i x_j)(x_i t^2 - x_j).$$

Then $Z_\rho(\boldsymbol{x};\boldsymbol{t})$, with subscripts back on our t_i's, is a generalization of this factorization. From the first few computable examples, we conjecture, and later prove, the following form.

Proposition 3.5.

$$Z_\rho(\boldsymbol{x};\boldsymbol{t}) = \prod_{i=1}^{r}(x_i - t_i) \prod_{1 \leq i < j \leq r} (x_i x_j - t_i t_j)(x_j - x_i t_i t_j). \tag{3.5}$$

Our goal for this chapter is to state a generalization of (3.5) valid for all λ and offer a proof using a Yang-Baxter equation for half-turn ice. The Yang-Baxter equation will precisely describe the effect symmetries of λ-half-turn ice have on the factorization of the partition function Z_λ.

3.3 The star-triangle relation

The first symmetry we look for in our partition function is one to describe the relationship among the r indices of variables x and t. We study this symmetry using the group action of S_r, the symmetric group on r letters, on polynomials in $\mathbb{C}[x_1, \ldots, x_r, t_1, \ldots, t_r]$. The action of $\sigma \in S_r$ on $f \in \mathbb{C}[x_1, \ldots, x_r, t_1, \ldots, t_r]$ is

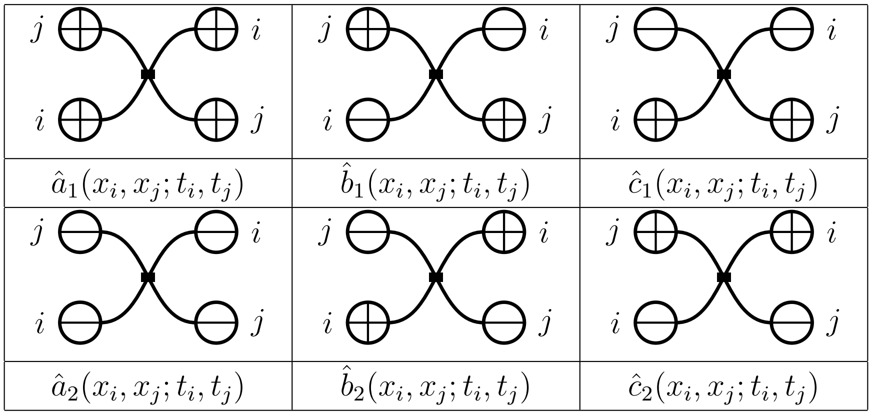
$$(\sigma \circ f)(x_1, \ldots, x_r; t_1, \ldots, t_r) = f(x_{\sigma(1)}, \ldots, x_{\sigma(r)}; t_{\sigma(1)}, \ldots, t_{\sigma(r)}), \tag{3.6}$$

the identical and simultaneous permutation of the indices of $\boldsymbol{x}$ and $\boldsymbol{t}$. We call a polynomial f *bisymmetric* if it is invariant under this action, i.e. $\sigma \circ f = f \;\; \forall \sigma \in S_r$.

The index of x and t in the Boltzmann weight of a vertex comes from its row positioning. Thus, the ice interpretation of this S_r action is the switching of the indices of the rows. In order to employ the local structure of square ice, we start by studying the actions of adjacent transpositions, the set of permutations $\{(i \;\; i+1) | 1 \leq i \leq r-1\}$ (writing the elements of S_r in cycle notation). This subset of S_r generates the entire group, so we may reduce the problem of the general action of $\sigma \in S_r$ to the action of adjacent transpositions.

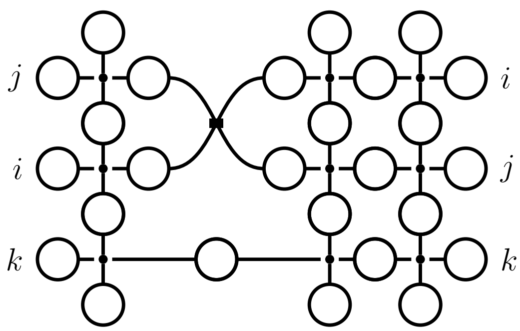
The action of each adjacent transposition on Z_λ corresponds to switching the labels of two adjacent rows (since the rows bend around, we are in fact switching the labels of two pairs of adjacent rows, which are linked by the U-shaped vertices on the right).

Twisted ice Twisted ice are special ice vertices rotated by 45 degrees counterclockwise and inserted into pieces of ice. Their purpose is to demonstrate symmetries of the partition function of the piece of ice. The six allowed fillings of twisted ice vertices and their (as yet undetermined) Boltzmann weights are in the following table.



(3.7)

These Boltzmann weights are treated the same as for normal square ice. Given a choice of Boltzmann weights for twisted ice (the six hatted polynomials in (3.7)), a partition function for a piece of ice including twisted ice vertices may be defined and evaluated just as before. One difference is that now a weight of a single vertex involves two separate indices, i and j. This is due to the way that twisted ice fits into a normal piece of ice, which is illustrated in (3.8). These particular notations come from [5] (with inspiration from [2]).



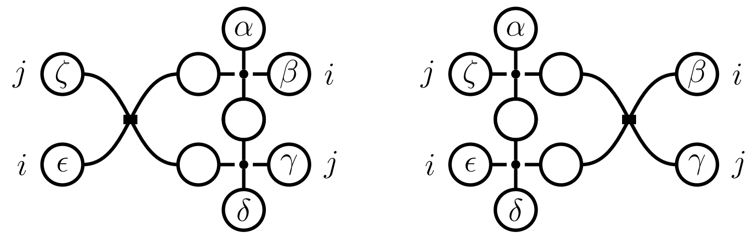
(3.8)

Thoughts to take away from (3.8) are below.

- All four edges attached to a twisted ice vertex connect horizontally to regular square ice.
- A piece of twisted ice fits between adjacent columns, and lies in two adjacent rows simultaneously. This is why the Boltzmann weight of a twisted ice vertex is a rational function on variables with two indices.
- If the rows to the left of the twisted ice are labeled with j over i, to the right of the twisted ice the labels are swapped: i is over j.

So if we can pass a piece of twisted ice across a column, we will have switched the labels of the two intervening vertices. This suggests how we can locally address the problem of swapping the indices of entire rows. The star-triangle relation exhibits the identity that will allow a twisted ice vertex to cross a column.

Lemma 3.6 (Star-triangle relation)**.** *There exists a choice of Boltzmann weights for twisted ice such that the partition functions for the two pieces of ice in* (3.9) *are equal, for any choice of boundary signs* $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \{+, -\}$.



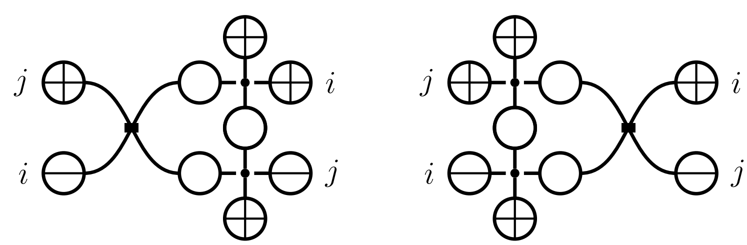
(3.9)

Remark 3.7. We have stated this in terms of black ice. It is also true for clear ice, with two separate sets of Boltzmann weights for twisted black ice and twisted clear ice.

Proof. Remember that these partition functions are the sums over the allowed fillings of the product of Boltzmann weights of each filling. For each choice of signs on the boundary, we write down a linear equation for the weights of the twisted ice vertices. Let us look at these linear equations in a bit of detail.

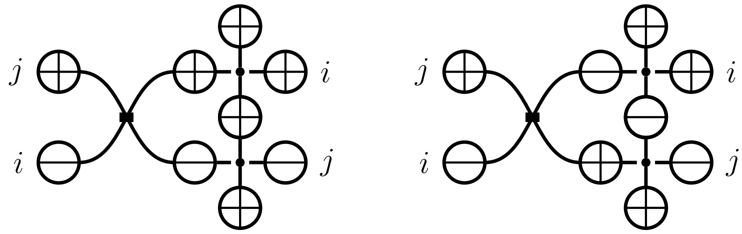
Observe that in both ice pieces flowlines enter the piece through α, ζ, and ϵ and exit out the other three edges. So the two multisets of signs $\{\alpha, \zeta, \epsilon\}$ and $\{\beta, \gamma, \delta\}$ are equal (i.e. they have the same number of $-$'s). This reduces the number of linear equations from $2^6 = 64$ to a somewhat more manageable 20.

We demonstrate the derivation of one of these equations. Let the choice of boundary signs be $+, +, -, +, -, +$, so we have the two pieces of ice in (3.10).



(3.10)

To compute the partition function for the left piece of ice, we find all its allowed fillings. There are two:

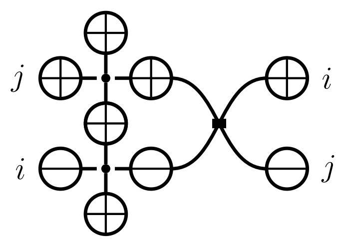


(3.11)

Therefore the partition function for the left piece of ice in (3.10) is the sum of the evaluations of the fillings (3.11),

$$\hat{c}_2(x_i, x_j; t_i, t_j) \cdot 1 \cdot x_j t_j \sqrt{-1} + \hat{b}_1(x_i, x_j; t_i, t_j) \cdot x_i(1 - t_i^2) \cdot 1.$$

The right piece of ice in (3.10) has only one allowable filling.



(3.12)

Therefore the partition function for the right piece of ice in (3.10) is simply the evaluation of (3.12),

$$\hat{c}_2(x_i, x_j; t_i, t_j) \cdot 1 \cdot x_i t_i \sqrt{-1}.$$

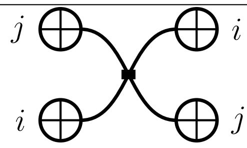
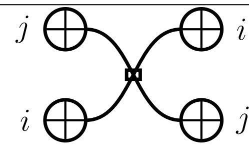
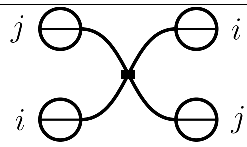
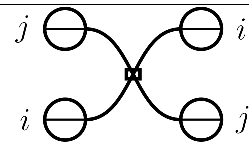
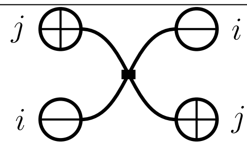
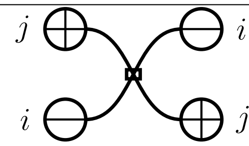
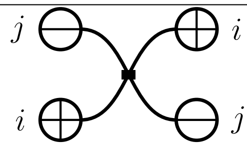
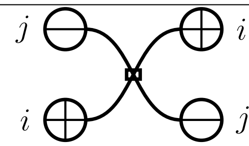
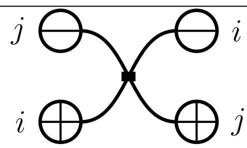
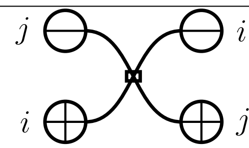
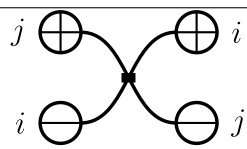
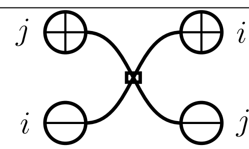
Equating these partition functions gives the following relation between the rational functions $\hat{b}_1$ and $\hat{c}_2$.

$$\hat{c}_2(x_i, x_j; t_i, t_j) \cdot x_j t_j \sqrt{-1} + \hat{b}_1(x_i, x_j; t_i, t_j) \cdot x_i(1 - t_i^2) = \hat{c}_2(x_i, x_j; t_i, t_j) \cdot x_i t_i \sqrt{-1}.$$

So we have eliminated one of the six "unknowns" (Boltzmann weights of twisted ice) of the system of equations, with (3.13):

$$\hat{b}_1(x_i, x_j; t_i, t_j) = \frac{(x_i t_i - x_j t_j)\sqrt{-1}}{x_i(1 - t_i^2)} \hat{c}_2(x_i, x_j; t_i, t_j). \tag{3.13}$$

For the system to have a nontrivial solution, a number of redundancies would be required, based on the number of equations (twenty) and variables (the six Boltzmann weights of twisted ice). This turns out to be the case, however, and the weights (determined by hand by solving this set of linear equations generated by the method that produced (3.13)) are in the following table. ■

Twisted black ice	Weight	Twisted clear ice	Weight
	$x_j - x_i t_i t_j$		$x_i - x_j t_i t_j$
	$x_i - x_j t_i t_j$		$x_j - x_i t_i t_j$
	$(x_i t_i - x_j t_j)\sqrt{-1}$		$(x_j t_i - x_i t_j)\sqrt{-1}$
	$(x_j t_i - x_i t_j)\sqrt{-1}$		$(x_i t_i - x_j t_j)\sqrt{-1}$
	$x_i(1 - t_i^2)$		$x_j(1 - t_j^2)$
	$x_j(1 - t_j^2)$		$x_i(1 - t_i^2)$

(3.14)

Notice that we once again have both clear and black twisted ice, which will later be used to introduce a twist to either the top or bottom halves of our square ice.

3.4 The train argument

Once we have the star-triangle relation, we can use it (in the method of [5] for square ice) to translate a piece of twisted ice horizontally across half-turn ice. The crucial observation is that the global partition function of the (entire piece of) half-turn ice in not affected.

We can see this by separating the piece of half-turn ice into two regions. The first region is made up of the twisted ice vertex and its two neighboring vertices to the right, which is the shape of the LHS of the star-triangle relation (3.9). The second region consists of the rest of the half-turn ice. We can then write the partition function of the entire piece of ice as a sum over assignments of $+$ and $-$ to the six boundary edges separating these two regions. Each summand is the product of the two partition functions of the two parts of ice with the given assignment to the boundary. By Lemma 3.6, we know that in each summand (and thus for fixed boundary values), the partition function for the first region can be replaced with the partition function for the RHS of the star-triangle relation with the same boundary values. Of course, the sum then becomes the partition function of the entire piece of ice with the twist moved past one column.

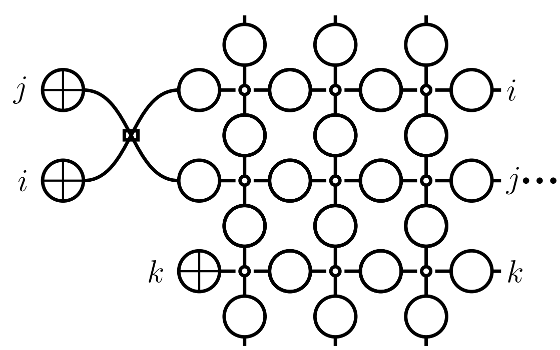
So we can translate a piece of twisted ice across columns. This leaves three places which require additional attention to complete the train argument.

1. We must study the initial effect on the partition function of introducing a twisted ice vertex into the piece of half-turn ice.

2. We need specialized arguments beyond the star-triangle identity to handle the effect of passing a twisted ice vertex past the half-turn right boundary.

3. Finally, we must analyze the effect of removing the twisted ice vertex from the half-turn ice, which returns the partition function for the (untwisted) half-turn ice with two adjacent rows swapped.

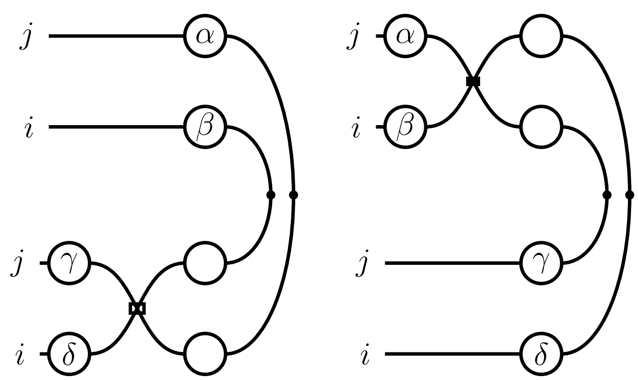
We would like to insert a piece of twisted ice into the square ice pattern in a way that minimally alters the partition function. Thus we insert the twist at the place we have the most control over the signs of the edges: the left boundary of the pattern. If we insert the twist on adjacent rows indexed by i and j, in clear ice, we get the following picture (note this picture only shows cropped view of the half-turn ice).



Let $Z(\boldsymbol{x};\boldsymbol{t})$ be the partition function of the half-turn ice piece before the twisted vertex is attached, and $\hat{Z}(\boldsymbol{x};\boldsymbol{t})$ be the partition function of the piece with the twist added. Since the left two edges of the twisted vertex are assigned $+$'s, the only allowed assignment of its right two edges are also +'s. Furthermore, once these +'s are entered, the rest of the ice is exactly the original half-turn ice whose partition function is $Z(\boldsymbol{x};\boldsymbol{t})$. This argument gives the equation

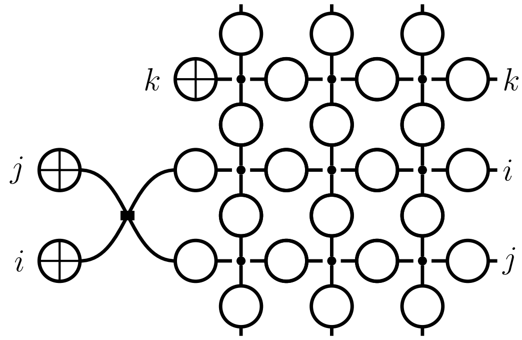
$$\hat{Z}(\boldsymbol{x};\boldsymbol{t}) = (x_i - x_j t_i t_j) \cdot Z(\boldsymbol{x};\boldsymbol{t}). \tag{3.15}$$

Now the star-triangle relation can be employed to move the twisted vertex to the right through the half-turn ice, switching the labels i and j as it moves rightward, while the partition function of the ice remains equal to $\hat{Z}$. After n applications of the star-triangle relation, we end up with the twist bordering the U-shaped vertices on the right boundary. Now we arrive at the second point requiring attention, we need to get the twisted ice around the bend. We compare the two partition functions for the following two pieces of ice, for a fixed list of + and − signs $\{\alpha, \beta, \gamma, \delta\}$.



Once again we only need to trouble ourselves with the cases for which $\{\alpha, \beta, \gamma, \delta\}$ give at least one allowed filling, which means that there are exactly two +'s and two −'s in the set. There are six equations to check, and it turns out that for all six the partition functions for the two pieces of ice are equal. So the clear ice twist can be pulled around the U-shaped pieces and it becomes a black ice twist.

We once again use the star-triangle another n times to work it back across the half-turn ice. As it goes, it switches the places of i and j on the black ice. After n applications, it finally rests to the left of all the ice again.



(3.16)

Now we are able to remove the twisted ice from the pattern by the same logic that we added the twist to the clear ice. The only allowed filling of the right two edges of the twist are both +'s, so we have that the Boltzmann weight of the black ice twist factors out:

$$\hat{Z}(\boldsymbol{x};\boldsymbol{t}) = (x_j - x_i t_i t_j) \cdot (i\ j) \circ [Z(\boldsymbol{x};\boldsymbol{t})]. \tag{3.17}$$

The transposition $(i\ j)$ acts on Z by swapping the indices i and j on both x and t.

Equating the two formulas for $\hat{Z}$, (3.15) and (3.17), we get

$$(x_i - x_j t_i t_j) Z(\boldsymbol{x};\boldsymbol{t}) = (x_j - x_i t_i t_j) \cdot (i\ j) \circ [Z(\boldsymbol{x};\boldsymbol{t})] = (i\ j) \circ [(x_i - x_j t_i t_j) Z(\boldsymbol{x};\boldsymbol{t})]. \tag{3.18}$$

Thus the product on the LHS of Equation (3.18) is invariant under the transposition $(i\ j)$. The following argument then implies that the product

$$\left(\prod_{1 \leq i < j \leq r} (x_i - x_j t_i t_j) \right) Z(\boldsymbol{x};\boldsymbol{t}) \tag{3.19}$$

is invariant under the transposition $(k\ k+1)$ for every $1 \leq k \leq r-1$. For any fixed such k, we factor the product (3.19) as

$$\left(\prod_{\substack{1 \leq i < j \leq r \\ (i,j) \neq (k,k+1)}} (x_i - x_j t_i t_j) \right) \cdot ((x_k - x_{k+1} t_k t_{k+1}) Z(\boldsymbol{x};\boldsymbol{t}))\,.$$

Both factors are invariant under the action of $(k\ k+1)$, so the product is as well. Since the product is simultaneously invariant under every member of a generating set of S_r, it is invariant under all of S_r. The action of each element of S_r permutes the indices of both x and t equally, hence the name *bisymmetry*. We have shown the following.

Proposition 3.8. *For any distinct partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, the product*

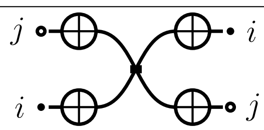
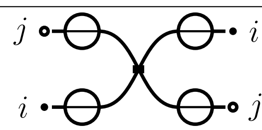
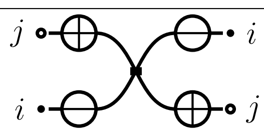
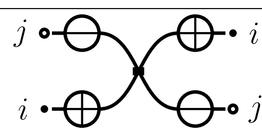
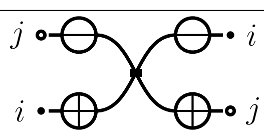
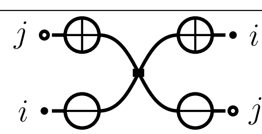
$$\left(\prod_{1 \leq i < j \leq r} (x_i - x_j t_i t_j) \right) Z_\lambda(\boldsymbol{x}; \boldsymbol{t})$$

is unchanged by simultaneous and equal rearrangements of the indices of $\boldsymbol{x}$ and $\boldsymbol{t}$, i.e., it is bisymmetric.

3.5 Gray ice

The train argument above gives a symmetry of the partition function by observing how it behaves upon switching adjacent rows of the ice pattern. The adjacent rows were assumed to be any two that are in the bottom half of the ice pattern, and resulted in also switching the corresponding adjacent rows in the top half of the pattern. However, we are still left with one pair of adjacent rows that we do not know how to switch: the middle pair of rows in the half-turn ice. This is because these two rows actually have different weights. To observe this additional symmetry we employ *gray ice*, which is twisted ice that twists a row of black ice with a row of clear ice.

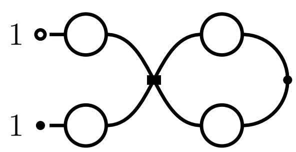
Just as in the above case, we want a star-triangle relation to hold so that the twist can propagate through the pattern without changing the partition function. To choose the appropriate Boltzmann weights, we solve a similar set of 20 linear equations to those for twisted black and clear ice. The results are in the table below.

Gray ice piece	Weight	Gray ice piece	Weight
	$1 - x_i x_j t_i t_j$		$x_i x_j - t_i t_j$
	$(x_i x_j t_i - t_j)\sqrt{-1}$		$(t_i - x_i x_j t_j)\sqrt{-1}$
	$x_j(1 - t_j^2)$		$x_i(1 - t_i^2)$

As we did with twisted ice, we start with a piece of half-turn ice with partition function $Z(\boldsymbol{x}, \boldsymbol{t})$. We then attach the gray ice to the far left of the innermost rows that are both indexed 1. The partition function for the piece of ice with this new twisted vertex we will call $\tilde{Z}(\boldsymbol{x};\boldsymbol{t})$. By an identical argument to the one that gave us (3.15), we have

$$\tilde{Z}(\boldsymbol{x};\boldsymbol{t}) = (1 - x_1^2 t_1^2) \cdot Z(\boldsymbol{x};\boldsymbol{t}). \tag{3.20}$$

Once the gray ice vertex is attached, we send it to the right through the half-turn pattern. As the gray ice vertex crosses the pattern it switches the black ice from top to bottom, and clear ice from bottom to top. By the star-triangle relation, the partition function remains unchanged, and so we can slide the twist all the way to the right of the pattern. Now, because these two rows are the middle rows, the picture is strictly different from the twisted ice case studied above.



(3.21)

Untwisting this piece of ice would create another symmetry of Z. So we compare two partition functions: the one of the untwisted U-shaped ice alone and the one of the pictured gray ice vertex and U-shaped ice. For each of the two boundary choices for these pieces of ice, the untwisted versions have only the trivial filling and the twisted version has two fillings. The partition functions are in the table below.

Boundary	Untwisted	Twisted
$1 \circ\!-\!\ominus$ $1 \bullet\!-\!\oplus$	1	$(x_1 - t_1)(1 + x_1 t_1)$
$1 \circ\!-\!\oplus$ $1 \bullet\!-\!\ominus$	$\sqrt{-1}$	$(x_1 - t_1)(1 + x_1 t_1)\sqrt{-1}$

We find that for the two boundaries, the partition function of the ice with the extra gray ice vertex is the same multiple of the untwisted Boltzmann weight of the half-turn ice. This means that we can factor this multiple out of $\tilde{Z}$ and get the untwisted partition function of the pattern with the black ice and clear ice switched in the rows indexed 1 (we can denote this partition function $Z^{(1)}$). So we have

$$\tilde{Z}(\boldsymbol{x};\boldsymbol{t}) = ((x_1 - t_1)(1 + x_1 t_1)) \cdot Z^{(1)}(\boldsymbol{x};\boldsymbol{t}). \tag{3.22}$$

Now we relate $Z^{(1)}$ to Z by computing the effect of switching the black and clear ice of the two middle rows. By studying the two tables of Boltzmann weights, we find that the only difference between the weights of a piece of black ice and a piece of clear ice is whether a power of x_1 appears. Since the weights are all multiplied together, the only visible difference between these is the total power of x_1.

The contribution of the row of clear ice to the exponent of x_1 is the number $+$'s assigned to horizontal edges that aren't the leftmost one, and for black ice, it is the number of $-$'s not counting the rightmost edge. Swapping the types of ice counts exactly the opposite signs, so we get that if the power of x_1 in the filling before the black ice and clear ice are swapped was x_1^k, then the power after the switch is precisely x_1^{2n-1-k}.

Since this holds for all possible fillings of our fixed boundary, the partition functions Z and $Z^{(1)}$ are related by the equation

$$Z^{(1)}(x_1, x_2, \ldots, x_r; t_1, t_2, \ldots, t_r) = x_1^{2n-1} \cdot Z(x_1^{-1}, x_2, \ldots, x_r; t_1, t_2, \ldots, t_r). \tag{3.23}$$

We now can combine the three equations involving Z, $\tilde{Z}$, and $Z^{(1)}$ ((3.20), (3.22), and (3.23)) to get a relation for our original partition function Z:

$$\begin{aligned}(1 - x_1^2 t_1^2) \cdot Z(\boldsymbol{x}; \boldsymbol{t}) &= \tilde{Z}(\boldsymbol{x}; \boldsymbol{t}) \\ &= (x_1 - t_1)(1 + x_1 t_1) \cdot Z^{(1)}(\boldsymbol{x}; \boldsymbol{t}) \\ &= (x_1 - t_1)(1 + x_1 t_1) \cdot x_1^{2n-1} \cdot Z(x_1^{-1}, x_2, \ldots, x_r; \boldsymbol{t}).\end{aligned}$$

Dividing by $(1 + x_1 t_1)$, we find that

$$(1 - x_1 t_1) \cdot Z(\boldsymbol{x}; \boldsymbol{t}) = (x_1 - t_1) \cdot x_1^{2n-1} Z(x_1^{-1}, x_2, \ldots, x_r; \boldsymbol{t}). \tag{3.24}$$

So we have found a new functional equation for the partition function Z.

It is useful to consider this symmetry in more abstract terms. We let the operator τ_i, for some $1 \leq i \leq m$, take a polynomial $f(x_1, x_2, \ldots, x_m)$ to the polynomial

$$x_i^{s+s'} \cdot f(x_1, x_2, \ldots, x_i^{-1}, \ldots, x_m),$$

for s the highest degree of x_i in f, and s' the lowest degree of x_i in f (i.e., s' is the highest power of x_i dividing f). We will call a polynomial fixed by τ_i *palindromic* in x_i.

We can easily observe that the τ_i are involutions, because they reflect the powers of x_i about its 'average' power, $(s+s')/2$, in each term of the polynomial. The τ_i are also multiplicative: for two polynomials $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$,

$$\tau_i \circ (fg) = (\tau_i \circ f) \cdot (\tau_i \circ g).$$

Since we have that

$$\tau_1 \circ (1 - x_1 t_1) = (x_1 - t_1),$$

we see from Equation (3.24) that $(1 - x_1 t_1) \cdot Z$ is palindromic in x_1. So we have proven the following proposition.

Proposition 3.9. *For any distinct partition λ, the product*

$$(1 - x_1 t_1) \cdot Z_\lambda(\boldsymbol{x}; \boldsymbol{t})$$

is invariant under τ_1, i.e., it is palindromic in x_1.

3.6 Factoring the Weyl denominator

We now show that these symmetries of the partition function actually imply that the deformation of the Weyl denominator on the RHS of Equation (3.5) factors out of Z_λ for any λ of length r.

Theorem 3.10. *We may write the partition function Z_λ of half-turn ice with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ as*

$$Z_\lambda(\boldsymbol{x}; \boldsymbol{t}) = \left[\prod_{i=1}^{r} (x_i - t_i)\right] \cdot \left[\prod_{1 \le j < k \le r} (x_k - x_j t_j t_k)(x_j x_k - t_j t_k)\right] \cdot \phi_\lambda(\boldsymbol{x}; \boldsymbol{t}), \tag{3.25}$$

for ϕ_λ a bisymmetric polynomial that is palindromic in every x_i.

Remark 3.11. We quickly note the meaning of these properties of ϕ_λ. We prove that this polynomial is invariant under the actions of S_r and τ_i for each i. The group that the actions $S_r \cup \{\tau_i\}$ generate is isomorphic to the Weyl group of $\mathrm{SO}_{2r+1}(\mathbb{C})$, a rank r type B Cartan group. For this reason, we call a polynomial that is invariant under both S_r and the τ_i *Weyl-invariant.*

Proof. We come to this conclusion as the result of the work done in Propositions 3.8

and 3.9. Let us consider the product

$$\prod_{1\le i<j\le r}(x_i - x_jt_it_j)\prod_{1\le i<j\le r}(1 - x_ix_jt_it_j)\prod_{i=1}^{r}(1 - x_it_i)Z_\lambda(\boldsymbol{x};\boldsymbol{t}). \tag{3.26}$$

If we group the leftmost product in (3.26) with the partition function, we have a bisymmetric polynomial by Proposition 3.8. By inspection, the middle and right products are each bisymmetric as well, so the entire product (3.26) is bisymmetric.

Also, by grouping the rightmost product in (3.26) with the partition function, we get a polynomial that is palindromic in x_1 by Proposition 3.9. Every term in the left product with an x_1 must have $i = 1$, since $i < j$, so is of the form $(x_1 - x_jt_1t_j)$. Then τ_1 sends this to the term of the middle product corresponding to the same indices. Since τ_1 is an involution, τ_1 just swaps the positions of these $2r - 2$ terms, so the entire product is both bisymmetric and palindromic in x_1. The bisymmetry further implies that (3.26) is in fact palindromic in x_i for all i, and thus Weyl-invariant.

We now use these properties to determine the factorization of the partition function in (3.25). By its bisymmetry and the fact that $(x_i - x_jt_it_j)$ divides (3.26) for all $i < j$, we must have that $(x_j - x_it_it_j)$ also divides the product. Since these terms are coprime to all three products, they must divide Z_λ. So we have that the product

$$\prod_{1\le i<j\le r}(x_j - x_it_it_j) \tag{3.27}$$

divides $Z_\lambda(\boldsymbol{x};\boldsymbol{t})$.

The product (3.26) is also palindromic in every x_i, so by the multiplicativity of τ_i, the set of factors of the product is closed under action by each τ_i. For any $i < j$, consider

$$\tau_j \circ (x_i - x_jt_it_j) = (x_ix_j - t_it_j).$$

This doesn't appear as a factor of any of the products, so must also divide the partition function (it is irreducible since $i \neq j$). Thus the product

$$\prod_{1\le i<j\le r}(x_ix_j - t_it_j) \tag{3.28}$$

divides $Z_\lambda(\boldsymbol{x};\boldsymbol{t})$.

Finally, the third product in (3.26) is invariant under S_r but not the τ_i. We can act upon $(1 - x_it_i)$ nontrivially only with τ_i, which gives $(x_i - t_i)$ as a factor of the product. This factor once again does not appear in the three products, so must divide the partition function. So the third product we have that divides the partition function is

$$\prod_{i=1}^{r}(x_i - t_i). \tag{3.29}$$

Since the three products (3.27), (3.28), and (3.29) all divide Z_λ and are pairwise coprime, we may define a polynomial $\phi_\lambda(\boldsymbol{x};\boldsymbol{t})$ which satisfies (3.25). Now we show that it is bisymmetric and palindromic in x_i for all i.

We consider (3.26), but now use Equation (3.25) to substitute for the partition function. This gives that the product

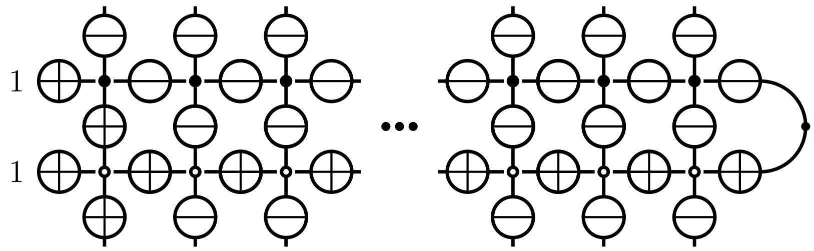
$$\prod_{\substack{i,j=1 \\ i\neq j}}^{r}(x_i - x_j t_i t_j) \prod_{1\le i<j\le r}[(x_i x_j - t_i t_j)(1 - x_i x_j t_i t_j)] \prod_{i=1}^{r}[(x_i - t_i)(1 - x_i t_i)]\,\phi(\boldsymbol{x};\boldsymbol{t}) \tag{3.30}$$

is bisymmetric and palindromic in all x_i. But each of the three products in (3.30) are bisymmetric, so ϕ_λ must also be bisymmetric. Also, each τ_i exchanges the $2r-2$ relevant terms of the first product with those of the second, and switches the order of the ith term in the third product. Therefore the entire polynomial which is being multiplied by ϕ_λ in (3.30) is both bisymmetric and palindromic in x_i for all i. Hence ϕ_λ inherits these properties, and we have completed the proof. ■

As a corollary, we explicitly compute the ρ partition function by a degree count.

Corollary 3.12. *For $\rho = (r, r-1, r-2, \ldots, 2, 1)$, the polynomial ϕ_ρ in Equation* (3.25) *is equal to 1.*

Proof. Let us consider Z_ρ as a polynomial in x_1. Each filling of the ρ boundary has total degree in x_1 determined by the Boltzmann weights in the middle two rows. If we want to find the maximal degree of x_1 in Z_ρ, we can limit our view to those two rows. The highest power of x_1 that we can get from a single vertex is 1, so we attempt to force every vertex in these two rows to contribute one x_1 to the product. Immediately we find that the leftmost vertex in the black ice row cannot contribute an x_1, since its left edge is always a $+$. Other than this vertex, we can arrange the signs of edges so that every other vertex contributes an x_1, for example, if the edges are assigned as in (3.31).



(3.31)

So the largest possible power of x_1 in Z_ρ is $2r-1$. Now we consider the factorization of Z_ρ, and see the degree of x_1 in each term.

$$Z_\rho(\boldsymbol{x};\boldsymbol{t}) = \underbrace{\prod_{i=1}^{r}(x_i - t_i)}_{x_1} \cdot \underbrace{\prod_{1\le j<k\le r}(x_k - x_j t_j t_k)}_{x_1^{r-1}} \cdot \underbrace{\prod_{1\le j<k\le r}(x_j x_k - t_j t_k)}_{x_1^{r-1}} \cdot \phi_\rho(\boldsymbol{x};\boldsymbol{t})$$

So the total degree of x_1 in the three products is $1+(r-1)+(r-1) = 2r-1$. Since this is the maximal possible degree of x_1 in Z_ρ, we know that ϕ_ρ must be of degree zero in x_1. By bisymmetry, ϕ_ρ has no terms with an x_i in them, and so is entirely a polynomial in $\boldsymbol{t}$.

Now we can argue similarly for the powers of t_1. We can first look at the Boltzmann weights of the vertices. This argument is trickier than the last, because each such Boltzmann weight can have t_1-degree of 0, 1, or 2, according to (3.4). However the only vertex filling with degree 2 has a $-$ to the left and a $+$ to the right, and so between any two of them must be a vertex filling with a $+$ sign to the left and $-$ sign to the right. The only vertex filling with this property has Boltzmann weight 1, and so the average weight of t_1 using these vertex fillings is at most 1. This suggests a maximal t_1-degree of $2r$, but by being careful we can reduce this to $2r-1$.

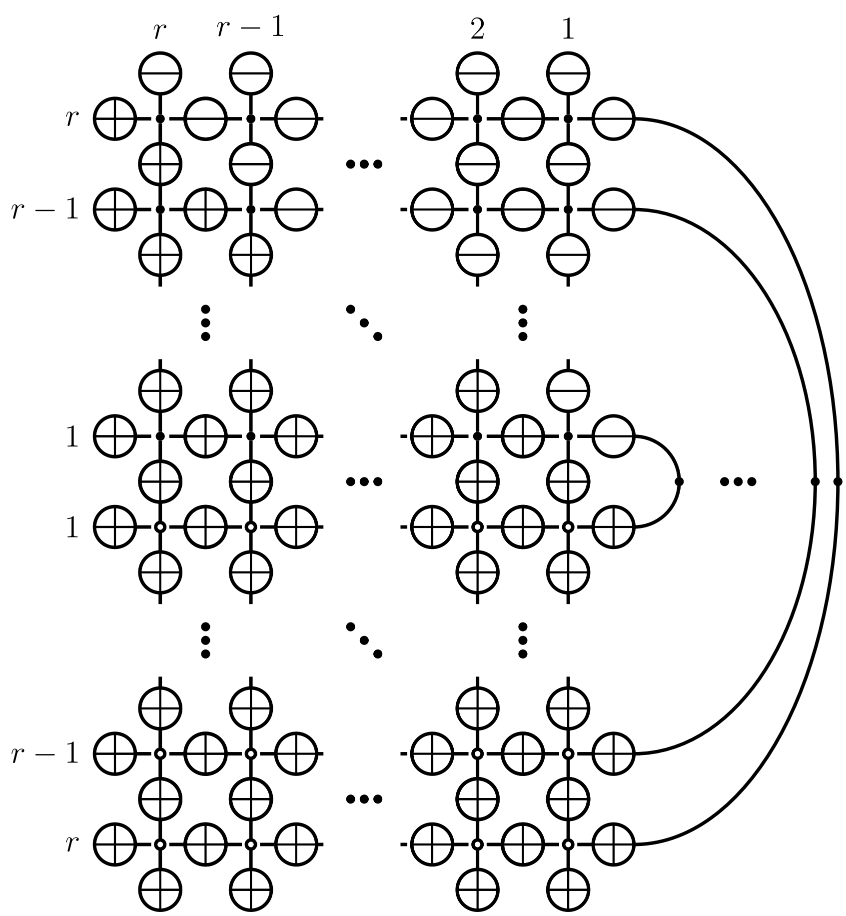
We know that the leftmost horizontal edges are both $+$'s, and the rightmost horizontal edges must differ in sign, since there is a half-turn piece of ice connecting them. Therefore one of the rows must have an odd number of vertices whose left and right edges are assigned different signs (the row that ends in a $-$). These vertices, read left to right, begin and end with the allowable vertex filling having with $+$ on the left and $-$ on the right, and so we observe that we have an extra vertex with weight 1, i.e. t_1-degree 0, which is not paired with a vertex with t_1-degree 2. Thus our maximal degree must be reduced by 1 to $2r-1$.

Now we can once again see that the degrees of the factors other than ϕ_ρ sum to this number:

$$Z_\lambda(\boldsymbol{x};\boldsymbol{t}) = \underbrace{\prod_{i=1}^{r}(x_i - t_i)}_{t_1} \cdot \underbrace{\prod_{1\le j<k\le r}(x_k - x_j t_j t_k)}_{t_1^{r-1}} \cdot \underbrace{\prod_{1\le i<j\le r}(x_j x_k - t_j t_k)}_{t_1^{r-1}} \cdot \phi_\lambda(\boldsymbol{x};\boldsymbol{t}).$$

So the total degree of t_1 in ϕ_ρ is 0. By bisymmetry, this implies that ϕ_ρ is a constant polynomial.

To compute this constant, observe that from (3.25) there is precisely one term of Z_ρ that is of t_i-degree 0 for all t_i. The coefficient of this term is the constant ϕ_ρ. So we catalog the fillings of the $\lambda = \rho$ boundary that do not contain the two vertex fillings with Boltzmann weights are divisible by t_i. The remaining four vertex fillings are exactly those that do not have their flowlines cross (see Remark 2.2), so the only possible filling is the one with zero flowline crossings, as in the following picture.



(3.32)

We can find the evaluation of the filling in (3.32) (and thus its contribution to Z_ρ) by finding the product of its vertices' Boltzmann weights row by row. From the black ice, the row corresponding to x_i has $i-1$ vertices with weight x_i and the other vertices have weight 1. Every clear ice vertex has weight x_i, so this gives r vertices with weight x_i for each i. Finally, the U-shaped ice pieces all have weight 1. So the total weight of this filling is

$$\prod_{i=1}^{r} x_i^{r+i-1}.$$

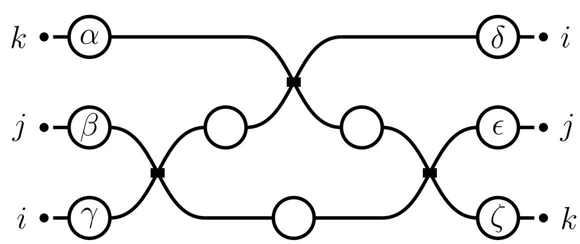
Since this term of Z_ρ has coefficient 1, the constant ϕ_ρ must be 1. ■

Chapter 4

Yang-Baxter equation

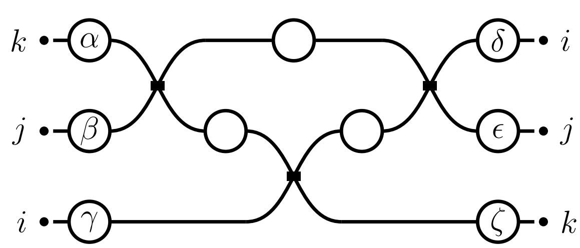
We have seen above that the star-triangle relation gives a very powerful local symmetry of the partition function of square ice. In particular, we can use it to compute the effect of swapping the indices of adjacent rows of the ice. One may then ask how this action interacts with itself. That is, with multiple twistings of rows, do additional symmetries of the partition function become apparent? This question is answered by the Yang-Baxter equation, a local symmetry of the partition function of *twisted* ice. The Yang-Baxter equation can be stated in a very similar form to the star-triangle relation.

Lemma 4.1 (Yang-Baxter equation)**.** *If* $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$ *is any list of* $+$ *or* $-$ *signs, then the partition functions of the two pieces of twisted ice with indexed vertices*



(4.1)

and



(4.2)

are equal.

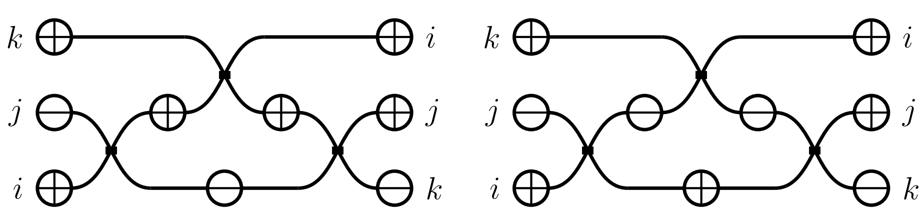
Remark 4.2. As was the case with the star-triangle relation, this lemma also holds for clear ice. We note that in the literature, both Lemmas 3.6 and 4.1 may be referred to as Yang-Baxter equations. Throughout this paper we will always differentiate them using the name 'star-triangle relation' for Lemma 3.6.

The Yang-Baxter equation is a braid relation between adjacent *twists* of the square ice pattern. This relation tells us that the order of twisting doesn't matter: it is only the final positions of the rows (or indices) that matter. In fact, we already could reason this implication from the train argument above, since ϕ_λ is in fact unchanged by switching indices, and the other factors of Z_λ depend only upon the ordering of the indices. So the new fact represented by the Yang-Baxter equation is that this symmetry occurs locally, rather than only globally. In essence, the star-triange relation gives a set of functional equations represented by the generators of S_r, but we only know that they symmetries of the equations behave according to the group law of S_r by the Yang-Baxter equation. Without the Yang-Baxter we would have a functional equation for every element of the free group generated by the $r-1$ generators of S_r, but Yang-Baxter tells us that these equations are redundant.

The proof of Yang-Baxter is again a series of 20 linear equations to be satisfied. For example, we have that the two boundaries given by

$$\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\} = \{+, -, +, +, +, -\}$$

are equal. The fillings for the first arrangement of twisted ice are twofold.

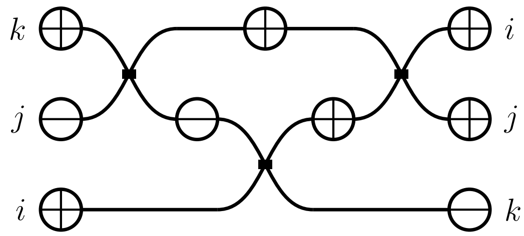


(4.3)

The local partition function for this arrangement is the sum of the evaluations of the fillings in Figure (4.3):

$$(x_j t_i - x_i t_j)i \cdot (x_k - x_i t_i t_k) \cdot x_k(1 - t_k^2) + x_i(1 - t_i^2) \cdot x_k(1 - t_k^2) \cdot (x_k t_j - x_j t_k)i$$
$$= (x_j - x_i t_i t_j) \cdot (x_k t_i - x_i t_k)i \cdot x_k(1 - t_k^2).$$

This final product is the evaluation of the only filling of the other arrangement of twisted ice with the same signs at the boundary pictured in Figure (4.4).



(4.4)

Chapter 5

A conjectural recursive equivalence

So far we have shown that each partition function Z_λ factors as the product of the deformation of the Weyl denominator and a Weyl-invariant polynomial ϕ_λ. In this chapter we attempt to compute ϕ_λ using a recursion. The recursion requires three parts: a base case and two independent recursive steps. These two steps specialize (by setting $t_i = 1$ for each i) to Pieri's Rule and Clebsch-Gordan theory, two known relations between characters of highest weight representations of type B. So given a proof of the recursion, a corollary would be that each ϕ_λ has special value the character of a highest weight representation of type B. However, even with the recursion, we would obtain an algorithm for computing the partition function, not a closed formula.

Remark 5.1. Throughout this chapter, we will refer to r, the length of λ, as the *rank*. This is due to the fact that the special value of ϕ_λ discussed above turns out to be the character of a representation of a group of rank r.

Also, we note that to correspond ϕ_λ (for distinct λ) with the character of a representation with highest weight μ (a not necessarily distinct partition), we have the following offset:

$$\lambda = \mu + \rho.$$

This substitution must be done for the following reformulations of Pieri's rule and Clebsch-Gordan theory.

Unfortunately, two of the three parts of the recursion have only partial proofs. The conjectured statements are below, and have been verified for all examples of ϕ_λ so far computed ($r \leq 3$ and $\lambda_1 \leq 10$). While we don't prove the recursion, in the final section of this chapter the special value at $t_i = 1$ is computed directly, and is shown to correspond to the expected type B character.

Before delving into the recursion, we define a convenient reformulation of ϕ_λ that will simplify the statements of the conjectures and propositions below.

Definition 5.2. For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ any distinct partition, and ϕ_λ the polynomial from (3.25), let $\psi_\lambda(\boldsymbol{x}; \boldsymbol{t})$ be the rational function defined by

$$\psi_\lambda(\boldsymbol{x}; \boldsymbol{t}) := \frac{\phi_\lambda(\boldsymbol{x}; \boldsymbol{t})}{u \prod_{i=1}^{r} x_i^{\lambda_1 - r}}. \tag{5.1}$$

where

$$u = \sqrt{-1}^{\sum \lambda_i - (r+1-i)}.$$

Remark 5.3. From our knowledge of ϕ_λ, we can infer that ψ_λ is bisymmetric. Also, by similar degree counting arguments to those found in Corollary 3.12, the largest degree of ϕ_λ in any x_i is $2 \cdot (\lambda_1 - r)$ and the smallest degree is zero. Therefore these values for ψ_λ become $(\lambda_1 - r)$ and $-(\lambda_1 - r)$. So ϕ_λ being palindromic in each x_i implies that ψ_λ is invariant under each of the r substitutions

$$x_i \longrightarrow x_i^{-1}.$$

Finally, the u factor will simplify the statement of Conjecture 5.5 and Theorem 5.15 and ensure that ψ is real. This factor will be ignored for the proof of Proposition 5.6, since it factors out of the equation to be proved, (5.4).

5.1 Base case

From Corollary 3.12 we obtain the first base case for any rank r, that $\psi_\rho = 1$ for each $\rho = (r, r-1, \ldots, 2, 1)$. In order to make use of our recursive relation, we will need more than just the ψ_ρ case. Hence we make the following conjecture for the form of ψ_λ for $\lambda = (a, r-1, r-2, \ldots, 2, 1)$ for arbitrary $a \geq r$. We will continue to refer to this as the base case.

Conjecture 5.4 (Base Case). *For $\lambda = (a, r-1, r-2, \ldots, 2, 1)$, with $a \geq r$, we have*

$$\psi_\lambda(\boldsymbol{x}; \boldsymbol{t}) = \sum_{\substack{i_1, \ldots, i_r \in \mathbb{Z} \\ \sum |i_k| \leq a-r}} \left[\left(\sum_{\substack{j_1, \ldots, j_r \in \mathbb{Z},\ j_k \geq |i_k| \forall k \\ a-r-1 \leq \sum j_l \leq a-r,\ j_k \equiv i_k\ (2) \forall k}} \left(\prod_{l=1}^{r} t_l^{j_l} \right) \right) \left(\prod_{k=1}^{r} x_k^{i_k} \right) \right]. \tag{5.2}$$

Letting $a = r$, we force all the i_k's and j_l's to be zero, and we are left with the known equality $\psi_\rho = 1$. For arbitrary a, if we specialize by setting $t_i = 1$ for all i, the inner sum has all summands equal to 1. So the inner sum becomes a count of the number of sets of indices j_l satisfying the requirements.

5.2 A deformation of Pieri's rule

One of the two recursions needed for our engine specializes to Pieri's rule for characters. The conjecture takes the following form.

Conjecture 5.5 (Deformation of Pieri's rule)**.** *For $\lambda = (\lambda_1, \ldots, \lambda_r)$ a distinct partition with $r > 1$, $\lambda_r = 1$, and $\lambda_2 = r$ when $r > 2$,*

$$\psi_\lambda(\boldsymbol{x};\boldsymbol{t}) \cdot \psi_{(r+1,r-1,r-2,\ldots,2,1)}(\boldsymbol{x};\boldsymbol{t}) = \sum_{\substack{\mu \text{ distinct} \\ \sum|\lambda_i - \mu_i| = 1}} \psi_\mu(\boldsymbol{x};\boldsymbol{t}). \tag{5.3}$$

The sum on the RHS of (5.3) is over length r distinct partitions μ that only differ from λ by exactly one at a single entry.

We have not yet found a proof of this step of the recursion. The method (from the proof of Lemma 3.6) of splitting the half-turn ice into two regions, a local one and a global one, has not been fruitful.

The way we employ this formula in our recursion is detailed below; first we introduce the other piece of the engine.

5.3 Clebsch-Gordan theory for the deformation

Clebsch-Gordan coefficients give formulas to factor the tensor product of two highest weight representations into the direct sum of irreducible representations. The character of a tensor product of representations is the product of the characters of the factors. Thus we can derive relations between characters via Clebsch-Gordan coefficients. Here we show that the collection of rational functions ψ_λ for varying λ also satisfy the same relations. We start with the rank two formulation.

Proposition 5.6 (Deformation of Clebsch-Gordan, Rank 2)**.** *Given $a > b > c > d > 0$,*

$$\psi_{(a,c)} \cdot \psi_{(b,d)} = \psi_{(a,d)} \cdot \psi_{(b,c)} + \psi_{(a,b)} \cdot \psi_{(c,d)}. \tag{5.4}$$

We will give a proof of this result, but first we state the more general cases. For arbitrary rank r, we have the following "lifted" version of Proposition 5.6.

Proposition 5.7. *Given $a > b > c > d > r-2$, we have in rank r that*

$$\psi_{(a,c,r-2,\ldots,1)} \cdot \psi_{(b,d,r-2,\ldots,1)} = \psi_{(a,d,r-2,\ldots,1)} \cdot \psi_{(b,c,r-2,\ldots,1)} + \psi_{(a,b,r-2,\ldots,1)} \cdot \psi_{(c,d,r-2,\ldots,1)}.$$

This follows by emulating the proof of 5.6. This result then gives rise to the most general form of this recursion:

Proposition 5.8 (Deformation of Clebsch-Gordan, General form)**.** *For $r > 1$, let $\lambda = (\lambda_1, \ldots, \lambda_{2r})$ be a partition with no more than two of any particular integer (i.e., $\lambda_i > \lambda_{i+2}$) and at least four unrepeated entries. Let $T \subseteq S_{2r}$ be the set of permutations σ that satisfy the following three conditions.*

1. *If λ_i is the largest entry of λ that is not repeated, then $\sigma^{-1}(i) \leq r$.*

2. *The r-tuples $(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(r)}) = \lambda^{(1)}(\sigma)$ and $(\lambda_{\sigma(r+1)}, \lambda_{\sigma(r+2)}, \ldots, \lambda_{\sigma(2r)}) = \lambda^{(2)}(\sigma)$ are both distinct partitions.*

3. *If $\lambda_i = \lambda_{i+1}$, then $\sigma^{-1}(i) \leq r$.*

Then

$$\sum_{\sigma \in T} \operatorname{sgn}(\sigma) \cdot \psi_{\lambda^{(1)}(\sigma)} \cdot \psi_{\lambda^{(2)}(\sigma)} = 0. \tag{5.5}$$

Before proving this result, we must introduce several new concepts regarding half-turn ice. But to connect the above three propositions, we first show how Proposition 5.6 is a special case of Proposition 5.8.

Letting $r = 2$ and $\lambda = (a, b, c, d)$ distinct in the hypothesis of Proposition 5.8, we compute that

$$T = \left\{1, \begin{pmatrix} 2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 3 \end{pmatrix}\right\} \subseteq S_4$$

in cycle notation. Therefore (5.5) becomes

$$\psi_{(a,b)} \cdot \psi_{(c,d)} - \psi_{(a,c)} \cdot \psi_{(b,d)} + \psi_{(a,d)} \cdot \psi_{(b,c)} = 0,$$

a restatement of Proposition 5.6. Similarly, for general r, taking $\lambda = (a, b, c, d, r-2, r-2, r-3, r-3, \ldots, 2, 2, 1, 1)$ specializes Proposition 5.8 to Proposition 5.7.

We prove Proposition 5.6 by appealing to the combinatorics of square ice. The following concepts are utilized in this proof.

ψ-Boltzmann weights Proposition 5.6 is stated in terms of ψ_λ. This polynomial is determined from λ-half-turn ice by first computing Z_λ, factoring out ϕ_λ, and finally dividing by the monomial in (5.1). Since we want to prove a result about the ψ_λ using arguments about the ice, it is in our interest to more directly connect them. We achieve this by altering our choice of Boltzmann weights.

If we multiply both sides of (5.1) by the deformation of the Weyl denominator in (3.5), we get

$$Z_\rho \cdot \psi_\lambda = \frac{Z_\lambda}{\prod_{i=1}^r x_i^{\lambda_1 - r}}, \tag{5.6}$$

noting that we may ignore the u factor in the definition of ψ for the entire Clebsch-Gordan argument (see Remark 5.3). This equation can be rewritten

$$\frac{Z_\rho}{\prod_{i=1}^r x_i^r} \cdot \psi_\lambda = \frac{Z_\lambda}{\prod_{i=1}^r x_i^{\lambda_1}}. \tag{5.7}$$

So we see that when working with ψ_λ, the more natural partition functions are the quotients

$$\frac{Z_\lambda}{\prod_{i=1}^r x_i^{\lambda_1}}. \tag{5.8}$$

For this to be a partition function, we need a choice of Boltzmann weights that would evaluate to it. This turns out to be a very simple change from our original choice of Boltzmann weights. A piece of λ-half-turn ice has λ_1 columns, and since there are two rows with each index i, there are exactly $2\lambda_1$ vertices associated to each x_i. Thus, by dividing each Boltzmann weight chosen above (both clear and black ice, but not U-shaped ice) by $x_i^{\frac{1}{2}}$, we get exactly (5.8) as our partition function. We will call this choice of Boltzmann weights the ψ-Boltzmann weights.

Infinite half-turn ice An additional benefit to using ψ-Boltzmann weights comes in the form of infinite half-turn ice.

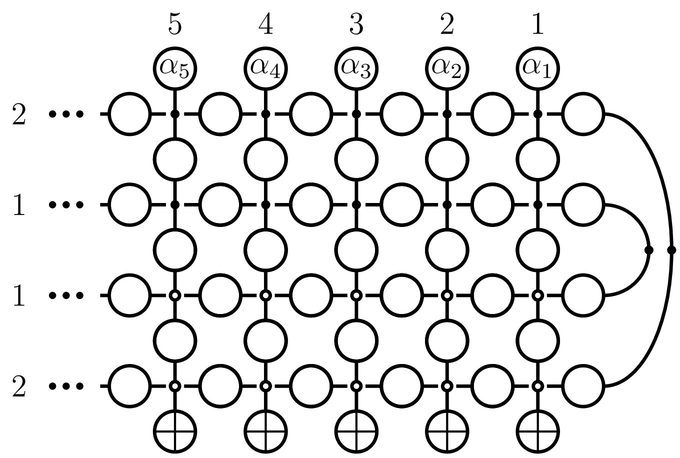
For some distinct partition λ, consider λ-half-turn ice. This has λ_1 columns, $2r$ rows, and a half-turn boundary on the right. Now, extend this piece of ice by tacking on a column of $2r$ vertices on the left. We still assign $+$'s and $-$'s to the boundary edges with the same formula: $+$'s are assigned to the left and bottom boundary edges, and $+$'s are assigned to the top boundary edges unless they lie in a column whose label is in λ.

However, now the leftmost column is labeled $\lambda_1 + 1$, so it is assigned a $+$ on the top. Because of this, any filling of this extended λ-half-turn ice is forced to assign $+$'s to edges adjacent to vertices in its leftmost column. That implies that the product of the weights may be factored out of the partition function of the extended piece of ice, and we are left with the normal λ-half-turn ice to be filled. This factor, the product of the $2r$ ψ-Boltzmann weights of the inserted column, is (from top to bottom)

$$z_r^{-\frac{1}{2}} \cdot z_{r-1}^{-\frac{1}{2}} \cdot \ldots \cdot z_1^{-\frac{1}{2}} \cdot z_1^{\frac{1}{2}} \cdot z_2^{\frac{1}{2}} \cdot \ldots \cdot z_r^{\frac{1}{2}} = 1.$$

Thus the extended piece of ice has an identical partition function.

Applying this principle repeatedly, we develop *infinite half-turn ice*. This is a piece of half-turn ice of some fixed rank r that has been extended infinitely to the left. The beauty of this new shape of half-turn ice is that every distinct partition λ of length r is associated to an assignment of boundary edge signs for the same ice shape.



(5.9)

The figure in (5.9) shows the form of infinite half-turn ice for rank two. Given any (a, b), a distinct partition of length two, we fix the top boundary values as $\alpha_a = \alpha_b = -$ and all other $\alpha_i = +$. Then the partition function for this piece of ice, using ψ-Boltzmann weights, will be

$$\frac{Z_{(a,b)}(x_1, x_2; t_1, t_2)}{x_1^a x_2^a},$$

the ψ-weight partition functions from (5.8).

Remark 5.9. We must clarify the definition of the partition function of an infinite sheet of ice. The difficulty lies in the infinite product over the vertices for each filling. We will use the ad hoc solution which takes the limit as $N \longrightarrow \infty$ of the partial product over the rightmost N columns of vertices. By the extension argument above, as long as there are exactly r $-$'s in the top boundary, this limit exists because the partial products stabilize for $N \geq \lambda_1$, the column of the leftmost $-$. So the limit equals the partition function generated by the ψ-Boltzmann weights (5.8).

If instead there are more than r $-$'s in the top boundary, there are no allowed fillings because only r negative flowlines can leave the piece of ice. If there are fewer than r $-$'s, the limit of the partial products does not exist because at least one negative flowline extends infinitely left, and this leads to an infinite power of some t_i.

Double half-turn ice Now we are ready to define double half-turn ice in the rank two case, which will be the combinatorial object used in the proof of Proposition 5.6. While this definition assumes we are working in rank two, its generalization to arbitrary rank is straightforward.

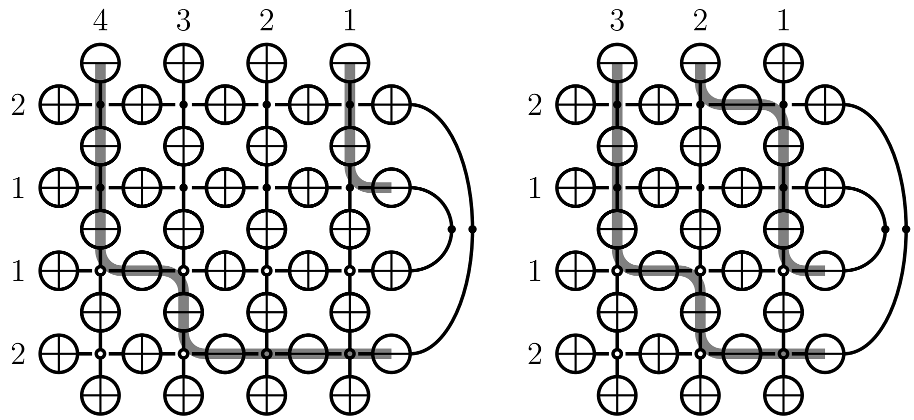
Definition 5.10. Given $a > b > c > d > 0$, we define (a, b, c, d)-*double half-turn ice*. The shape of this piece of ice is the same as rank two infinite half-turn ice, as in Figure (5.9). However, the boundary conditions are fundamentally different: in **double** half-turn ice, each boundary edge is assigned an unordered pair of signs ($++$, $+-$, or $--$). In the boundary conditions for (a, b, c, d)-double half-turn ice, the bottom boundary edges are all assigned $++$, the top boundary edges in columns not labeled a, b, c, or d are also assigned $++$, and the last four edges are assigned $+-$.

Fillings of double half-turn ice will be called *double fillings*, they consist of unordered pairs of fillings of the infinite half-turn ice such that the combination of signs on each boundary edge matches the boundary condition. So, if the boundary edge is assigned $++$, in both fillings that edge is assigned a $+$, similarly for $--$, and if the boundary is assigned $+-$, then the boundary edge is assigned $+$ in one of the fillings and $-$ in the other.

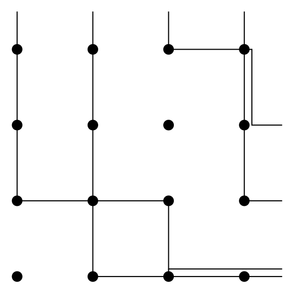
Note that both fillings of a double filling must be an allowable filling of the piece of infinite half-turn ice. Thus the right boundary must have U-shaped vertices with opposite signs on its edges, and every vertex must be one of the six allowed vertices in (2.1). Given any double filling T of a piece of double half-turn ice, define the *evaluation* of T to be the total product of all the Boltzmann weights of the vertices in the two fillings, in the limiting sense of Remark 5.9. So for (a, b, c, d)-double half-turn ice, the evaluation of any double filling occurs as a single summand one of the three products below (up to a factor of the deformed Weyl denominator).

$$\psi_{(a,b)}\psi_{(c,d)} \qquad \psi_{(a,c)}\psi_{(b,c)} \qquad \psi_{(a,d)}\psi_{(b,c)} \tag{5.10}$$

Silhouettes Given a double filling of a piece of double half-turn ice, we may consider the flowlines (see Remark 2.2) of each of its two fillings. In particular, we only draw the negative flowlines from each of the fillings, which contain all the information about the filling (any edge in a flowline is assigned a $-$, and the other edges are assigned a $+$). Now we overlap the two fillings onto one piece of infinite half-turn ice and superimpose the negative flowlines from both fillings to form the double filling's *silhouette*. An example of a double filling and its silhouette are shown in (5.11) and (5.12), on $(4,3,2,1)$-double half-turn ice.



(5.11)



(5.12)

We note several important features of silhouettes.

- A silhouette 'forgets' which flowline comes from which filling. More precisely, silhouettes even forget from which filling each segment of each flowline comes.

- Silhouettes do remember the number of negative flowlines along an edge (either 0, 1, or 2). Negative flowlines in a silhouette will be called *singular* or *double* at an edge depending on whether there are one or two negative flows that are superimposed onto it.

- The partition λ indexing a piece of double half-turn ice may be recovered from the a silhouette of a filling of that ice, by observing the numbers of columns where flowlines originate. This allows us to index a silhouette by the partition of its piece of double half-turn ice. The silhouette in (5.12) is indexed by the partition $(4,3,2,1)$.

- We will usually crop out the parts of the infinite half-turn ice whose silhouette is blank.

Silhouettes are useful because we can group double fillings which share the same silhouette. This allows us to compare the evaluations of double fillings contributing to the three products in (5.10), which are also the products that occur in the statement of Proposition 5.6. If we can show that the equation in the proposition, (5.4), holds for every collection of double fillings having the same silhouette, then summing these equations shows that the proposition holds. We will say that a silhouette is *satisfactory* when the contribution of the collection of double fillings that form that silhouette balance (5.4).

Given a rank two silhouette $\mathcal{S}$ indexed by the distinct partition (a, b, c, d), we define the function

$$\chi^{\mathcal{S}}_{(a,b)(c,d)}(\boldsymbol{x}; \boldsymbol{t})$$

to be the sum, over double fillings $\{T_{(a,b)}, T_{(c,d)}\}$ with silhouette $\mathcal{S}$ such that $T_{(a,b)}$ is a filling of (a, b)-half-turn ice, of the evaluation of the double filling. The two functions

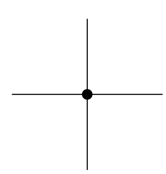
$$\chi^{\mathcal{S}}_{(a,c)(b,d)}(\boldsymbol{x}; \boldsymbol{t}) \text{ and } \chi^{\mathcal{S}}_{(a,d)(b,c)}(\boldsymbol{x}; \boldsymbol{t})$$

are defined similarly. Then a silhouette $\mathcal{S}$ is satisfactory exactly when

$$\chi^{\mathcal{S}}_{(a,c)(b,d)}(\boldsymbol{x}; \boldsymbol{t}) = \chi^{\mathcal{S}}_{(a,b)(c,d)}(\boldsymbol{x}; \boldsymbol{t}) + \chi^{\mathcal{S}}_{(a,d)(b,c)}(\boldsymbol{x}; \boldsymbol{t}). \tag{5.13}$$

Our strategy for the proof of Proposition 5.6 will be to show that all rank two silhouettes are satisfactory by induction. Before beginning the proof, we first prove a simple lemma concerning the vertices of silhouettes and describe the statistic over which we will induct.

Because we use silhouettes to group double fillings, we study the information lost about a double filling by only considering its silhouette. We check this at the vertex level, which means asking the following question. Given a single vertex in a silhouette, can we determine the two vertex fillings of the double filling which produced it? It turns out that the answer to this question is "Yes, in all but one case." The only ambiguous silhouetted vertex is the one with a singular negative flowline on all four adjacent edges, pictured in (5.14) below.



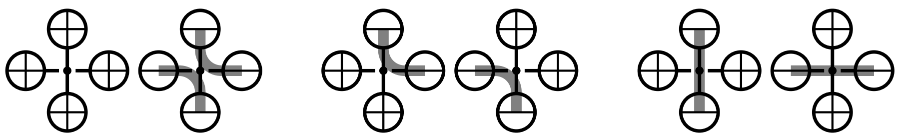
(5.14)

Lemma 5.11. *A vertex of a silhouette has a unique unordered pair of vertex fillings which create it unless it is of the form shown in* (5.14)*.*

Proof. If the vertex is not of the form in the figure, then it has at least one edge with 0 or 2 (negative) flowlines. Every flowline that enters the vertex must exit it, so the total number of flowlines on edges must be even. Then at least two of the edges have 0 or 2 flowlines. The signs of these two edges are determined in the double filling, since they are either both $+$ or both $-$.

If the other two edges each have a singular flowline, then arbitrarily choose one of them, and assign the first vertex a $+$ at that edge and the second vertex a $-$ at that edge. Then by a parity argument, the final edge can only be chosen in one way to create two allowed vertices. Finally, if one of the other two edges has 0 or 2 flowlines, both of them do and the two vertices that create them are fixed and identical by the argument in the last paragraph. ■

Remark 5.12. The silhouette vertex that is the exception to Lemma 5.11 can be split into the three distinct pairs of unordered vertex fillings in Figure (5.15).



(5.15)

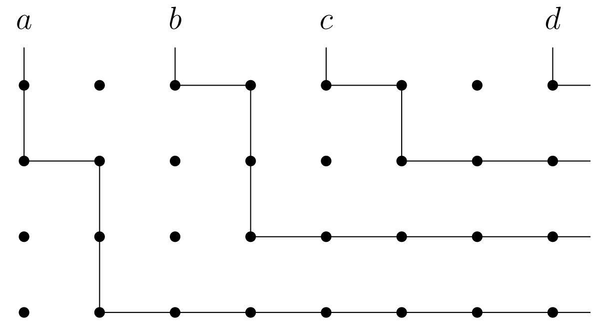
Collisions In a silhouette, a *collision* is a vertex where two negative flowlines meet, so there is at least one negative flowline coming into the vertex along its left edge and at least one along its top edge. Collisions come in two types. *Type 1* collisions occur when the "ambiguous" vertex (5.14) is formed. *Type 2* collisions include every other possible collision, those in which at least one edge of the vertex has a double flowline. The example of a silhouette given in (5.12) above has one Type 1 collision in the bottom left, and two Type 2 collisions to the right.

We finally prove Proposition 5.6 by showing that every silhouette is satisfactory. We use induction on the number of collisions in a silhouette.

Proof of Clebsch-Gordan, Rank 2. We begin with the following base case of our induction.

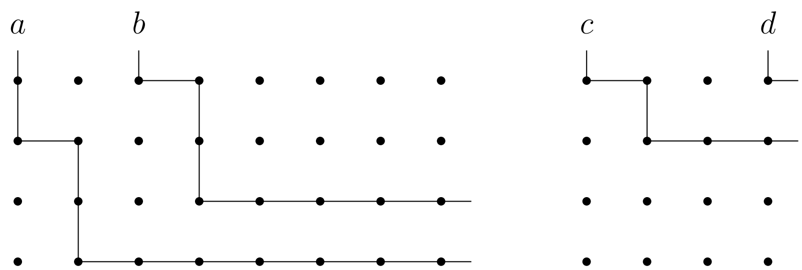
Lemma 5.13. *For $a > b > c > d > 0$, if a silhouette $\mathcal{S}$ indexed by (a, b, c, d) has no collisions, it is satisfactory.*

Proof. We consider such a silhouette. We will use (5.16) as an example of this general form.



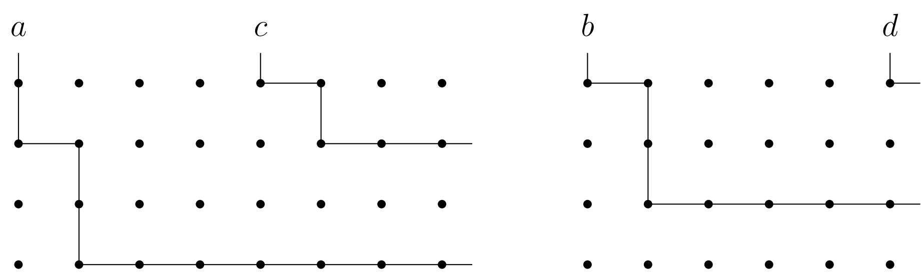
(5.16)

Now consider the silhouette's partition functions $\chi^{\mathcal{S}}_{(a,b)(c,d)}, \chi^{\mathcal{S}}_{(a,c)(b,d)}$, and $\chi^{\mathcal{S}}_{(a,d)(b,c)}$. By inspection, we find that exactly two double fillings have $\mathcal{S}$ as a silhouette. The first has evaluation contributing to $\chi^{\mathcal{S}}_{(a,b)(c,d)}$. It is the pair of fillings in (5.17).



(5.17)

The second double filling contributes to $\chi^{\mathcal{S}}_{(a,c)(b,d)}$ and splits into the following two fillings.



(5.18)

Because the silhouette has no collisions, in particular it has no Type 1 collision. Therefore, each vertex has a unique unordered pair of vertex fillings that superimpose to make it, by Lemma 5.11. This implies that the two double fillings above have identical evaluations. So we get the two equations

$$\chi^{\mathcal{S}}_{(a,b)(c,d)} = \chi^{\mathcal{S}}_{(a,c)(b,d)} \qquad\qquad \chi^{\mathcal{S}}_{(a,d)(b,c)} = 0.$$

Summing we get that $\mathcal{S}$ is satisfactory:

$$\chi^{\mathcal{S}}_{(a,c)(b,d)} = \chi^{\mathcal{S}}_{(a,b)(c,d)} + \chi^{\mathcal{S}}_{(a,d)(b,c)}.$$

This proves Lemma 5.13, the "base case" of our induction; note that it holds for an arbitrary distinct partition (a, b, c, d). ■

Now we state the inductive hypothesis. Assume that, for some fixed $k \geq 0$, all rank two silhouettes indexed by distinct partitions with at most k collisions are satisfactory.

Now we are given a rank two silhouette $\mathcal{S}$ indexed by (a, b, c, d) distinct with precisely $k+1$ collisions. Since $k+1 \geq 1$, there is at least one collision in $\mathcal{S}$. Of these collisions, find the one positioned furthest left. If there is a tie, choose the topmost of these collisions. We have two cases, depending on whether this collision is Type 1 or Type 2. Due to the simplicity of the Type 2 case, we will solve that first.

If this leftmost collision is Type 2, then it must be one of following three cases (where, in each case, we label flowlines according to the column from which they originate).

1. Flowline a enters the vertex from the left and b enters from the top, and they flow out along the same edge (either right or down).

2. Flowline b enters from the left and c enters from the top, and they flow out along the same edge.

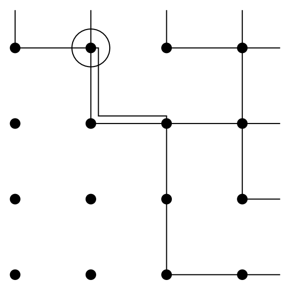
3. Flowline c enters from the left and d enters from the top, and they flow out along the same edge.

This can be deduced from the assumption that this is the "first" collision along the flowlines, which flow down and to the right. Because they flow out of the vertex along the same edge, any pair of fillings that results in this silhouette cannot pair the two flowlines involved together in the same filling.

Consider if the first case above occurs, so a and b flow in from the left and top edges of the vertex respectively. Now, we fix an arbitrary double filling which give this silhouette. This double filling contributes either to $\chi^{\mathcal{S}}_{(a,c)(b,d)}$ or $\chi^{\mathcal{S}}_{(a,d)(b,c)}$. The crucial observation is that whichever it is, we can simply swap the flows coming into this leftmost collision between the two fillings in our pair. This will switch the flowlines a and b, so it will swap the partition function to which it is contributing between the two possible ones above. These two partition functions of $\mathcal{S}$ are on different sides of (5.13), and the swap of flowlines does not involve any Type 1 collisions, so must preserve the evaluation. Therefore the double fillings with silhouette $\mathcal{S}$ may be paired up in such a way that shows that $\mathcal{S}$ is satisfactory.

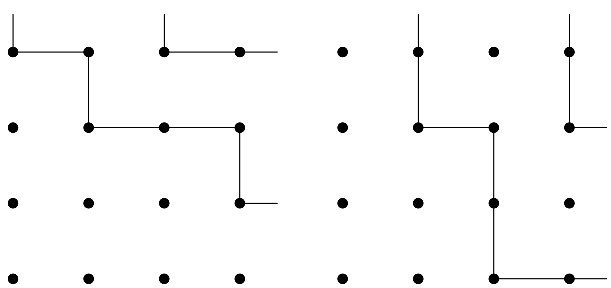
If instead we are in the second or third case listed above, the same argument is applied. If the two flowlines entering the vertex are b and c, then the two possible partition functions are those with subscript $(a, b)(c, d)$ or $(a, c)(b, d)$, which also are on opposite sides of (5.13). In the final case, the subscripts cannot pair c and d, and we are back to the same pair of partition functions as in the first case. Therefore in all cases, the double fillings may be paired up to show that $\mathcal{S}$ is satisfactory.

An example of this pairing of double fillings is illustrated below. We start with a silhouette



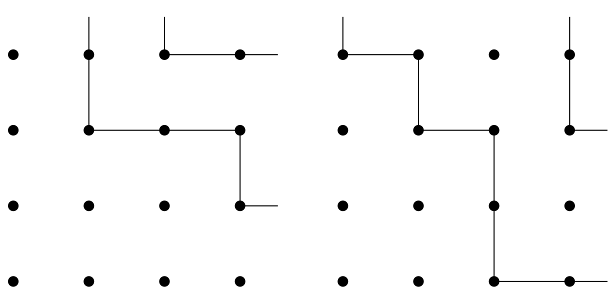
(5.19)

with the leftmost (Type 2) collision circled. Now we are given double filling with this silhouette, in (5.20).



(5.20)

We pair this double filling, which contributes to $\chi^{\mathcal{S}}_{(4,2)(3,1)}$, with the following double filling



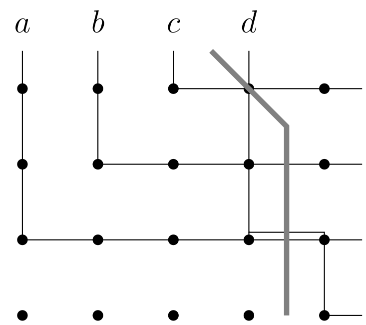
(5.21)

which contributes the exact same value to $\chi^{\mathcal{S}}_{(4,1)(3,2)}$, balancing the equation.

So now we must deal with the second case: if this leftmost collision is Type 1.

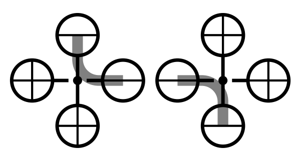
We will start with a silhouette $\mathcal{S}$ whose leftmost collision is Type 1, and we reduce to a silhouette with one fewer collision by separating this leftmost collision. This will enable us to apply the inductive hypothesis, and with careful bookkeeping we will have shown that $\mathcal{S}$ is satisfactory.

To separate the leftmost collision of $\mathcal{S}$, we insert a new column of vertex fillings directly beneath the collision vertex. This can be done by "cutting" from the bottom right of the collision's column in $\mathcal{S}$, straight up across the edges separating the collision's column and the column to its right. When the cut gets to the collision, which is Type 1, it turns diagonally left and cuts through the collision. An example of this cut is illustrated in the LHS of (5.22) below.



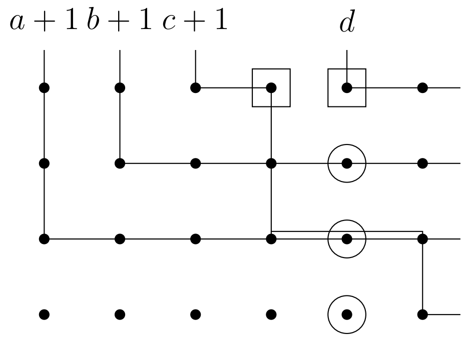
(5.22)

Once this cut is made, the flowlines of $\mathcal{S}$ are separated into two parts. So we can take the left half of the flowlines that were severed by the cut and pull them all rigidly to the left one vertex. The affected flowlines are exactly the flowline coming in from the left of the collision vertex, and all those that originated to the left of that one. So the Type 1 collision has been replaced by the two vertices in (5.23)



(5.23)

Now we fill the vacant vertices directly beneath where the collision was. These new vertices have horizontal flowlines only, rejoining the severed flowlines. So now we have replaced $\mathcal{S}$ with a new silhouette, $\mathcal{S}'$. We show $\mathcal{S}'$ in (5.24), with the two vertices which separated the collision boxed and the vertices which extended flowlines circled.



(5.24)

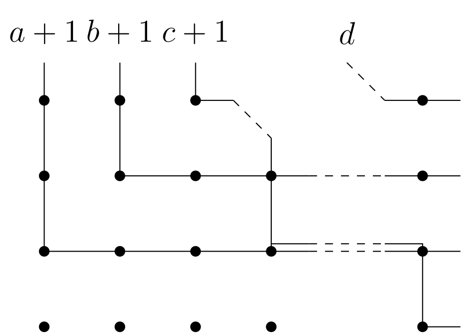
This new silhouette is indexed by a different partition, since the originating columns of some of the flows were moved left one, but the important fact is that $\mathcal{S}'$ has only k collisions. Therefore it is satisfactory, by the inductive hypothesis.

Since $\mathcal{S}'$ is satisfactory, we have the relation (5.25) among its partition functions. Following along with the picture, we use the partition $(a+1, b+1, c+1, d)$ to index $\mathcal{S}'$, even though it could also be $(a+1, b+1, c, d)$ or $(a+1, b, c, d)$.

$$\chi^{\mathcal{S}'}_{(a+1,c+1)(b+1,d)} = \chi^{\mathcal{S}'}_{(a+1,b+1)(c+1,d)} + \chi^{\mathcal{S}'}_{(a+1,d+1)(b+1,c)}. \tag{5.25}$$

Now we want to relate the partition function of the extended silhouette $\mathcal{S}'$ to the partition function of the original silhouette $\mathcal{S}$. Note that the extensions (the circled vertices in (5.24)) and the separation (the boxed vertices in (5.24)) that we added to $\mathcal{S}$ never contain a Type 1 collision. By Lemma 5.11, the Boltzmann weights of these vertices are fixed for every double filling and will factor out of all three partition functions for $\mathcal{S}'$.

Therefore we can divide (5.25) by this factor, essentially erasing these boxed and circled vertices from $\mathcal{S}'$. However, we must note that when two singular flowlines are joined by a singular flowline without collisions, they are forced to be in the same filling of a double filling. So when we erase the vertices, we need to remember these joinings, which can be drawn by dashed lines as in (5.26) below.



(5.26)

Now we can actually see that the satisfactory silhouette we have is essentially our original silhouette $\mathcal{S}$ with the collision vertex erased, but the flowlines that went through the collision connected in such a way that they do not cross. These dashed lines are important because they restrict how the silhouette can split up into two half-turn ice fillings.

We let $\mathcal{R}$ be the silhouette $\mathcal{S}$ with leftmost collision replaced with the dashed lines in (5.26). By the factoring argument, we have that $\mathcal{R}$ is satisfactory:

$$\chi^{\mathcal{R}}_{(a,c)(b,d)}(\boldsymbol{x};\boldsymbol{t}) = \chi^{\mathcal{R}}_{(a,b)(c,d)}(\boldsymbol{x};\boldsymbol{t}) + \chi^{\mathcal{R}}_{(a,d)(b,c)}(\boldsymbol{x};\boldsymbol{t}).$$

Now we relate $\chi^{\mathcal{S}}$ to $\chi^{\mathcal{R}}$. We need to break this into three cases depending on which flowlines are involved in the leftmost collision (which we will let be in a row indexed by i). First, we assume they are the a and b flowlines. Then we immediately get that

$$\chi^{\mathcal{S}}_{(a,b)(c,d)} = x_i^{\frac{1}{2}} \cdot x_i^{-\frac{1}{2}} \cdot \chi^{\mathcal{R}}_{(a,b)(c,d)} \tag{5.27}$$

because in this case, since a and b are paired together, the erased vertex gets filled in by a vertex with all $-$ edges in the (a,b) filling and a vertex with all $+$ edges in the (c,d) filling. One of these will have ψ-weight $x_i^{\frac{1}{2}}$ and the other $x_i^{-\frac{1}{2}}$ (although the order depends on whether it is black ice or clear ice).

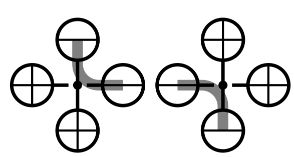
We next observe that

$$\chi^{\mathcal{S}}_{(a,c)(b,d)} = x_i^{-\frac{1}{2}} \cdot x_i^{\frac{1}{2}}(1-t_i^2) \cdot \chi^{\mathcal{R}}_{(a,c)(b,d)} + x_i^{-\frac{1}{2}} t_i i \cdot x_i^{\frac{1}{2}} t_i i \chi^{\mathcal{R}}_{(a,d)(b,c)} \tag{5.28}$$

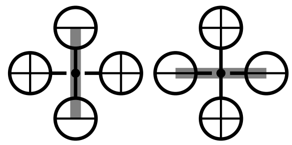
and

$$\chi^{\mathcal{S}}_{(a,d)(b,c)} = x_i^{-\frac{1}{2}} \cdot x_i^{\frac{1}{2}}(1-t_i^2) \cdot \chi^{\mathcal{R}}_{(a,d)(b,c)} + x_i^{-\frac{1}{2}} t_i i \cdot x_i^{\frac{1}{2}} t_i i \chi^{\mathcal{R}}_{(a,c)(b,d)}. \tag{5.29}$$

Here a is not paired with b, so there are two possible fillings of the erased vertex. Either the a flow can turn and flow downwards, and the b flow can turn and flow right, or they can both flow straight and cross each other (allowed, since they are unpaired). When they do not cross we get the first summand of the right hand side which does not switch the subscript, because in $\mathcal{R}$ it is assumed that a and b do not cross by the dashed lines. For this reason this first summand has product of the ψ-weights of the two vertex fillings



as its coefficient (neither of these depend on whether the ice is black or clear). When the two flows continue straight on in their respective fillings, and thus cross in the silhouette, we get the second summand of the right hand side. The coefficient of this summand is the product of the ψ-weights of the two filled in vertices shown here.



Here the subscript of χ changes because a and b cross each other.

Now, combining (5.27), (5.28), and (5.29) we get

$$\begin{aligned}
\chi^{\mathcal{S}}_{(a,c)(b,d)} &= (1-t_i^2)\cdot\chi^{\mathcal{R}}_{(a,c)(b,d)} - t_i^2\cdot\chi^{\mathcal{R}}_{(a,d)(b,c)} \\
&= \chi^{\mathcal{R}}_{(a,c)(b,d)} - t_i^2\cdot\chi^{\mathcal{R}}_{(a,c)(b,d)} - t_i^2\cdot\chi^{\mathcal{R}}_{(a,d)(b,c)} \\
&= (\chi^{\mathcal{R}}_{(a,b)(c,d)} + \chi^{\mathcal{R}}_{(a,d)(b,c)}) - t_i^2\cdot\chi^{\mathcal{R}}_{(a,c)(b,d)} - t_i^2\chi^{\mathcal{R}}_{(a,d)(b,c)} \\
&= \chi^{\mathcal{R}}_{(a,b)(c,d)} + (1-t_i^2)\cdot\chi^{\mathcal{R}}_{(a,d)(b,c)} - t_i^2\cdot\chi^{\mathcal{R}}_{(a,c)(b,d)} \\
&= \chi^{\mathcal{S}}_{(a,b)(c,d)} + \chi^{\mathcal{S}}_{(a,d)(b,c)}.
\end{aligned}$$

Thus $\mathcal{S}$ is satisfactory. The other two cases of the pairs of flowlines involved in the collision are resolved similarly. This completes our induction, and so all rank two silhouettes indexed by distinct partitions are satisfactory, which implies the Proposition. ■

5.4 The conjectural recursion

Now we can gather the conjectures stated and the results proven above to derive a general formula for ψ_λ. First, we write out the steps for a few small rank cases.

For rank one, the base case actually gives the full formula:

$$\psi_{(a)}(x;t) = \sum_{\substack{i\in\mathbb{Z} \\ |i|\le a-1}} \left[\left(\sum_{\substack{j\in\mathbb{Z}, i\equiv j(2) \\ a-2\le j\le a-1, |i|\le j}} \left(t^j\right) \right) \left(x^i\right) \right].$$

In rank two, we start with a general $\lambda = (a,b)$ with $b > 2$. Using the Clebsch-Gordan result:

$$\psi_{(a,b)} = \psi_{(a,2)}\psi_{(b,1)} - \psi_{(a,2)}\psi_{(b,1)}.$$

Then, if $b = 2$, we can instead use the Pieri's Rule result to get

$$\psi_{(a,2)} = \psi_{(a,1)}\psi_{(3,1)} - \psi_{(a+1,1)} - \psi_{(a-1,1)}.$$

So we have reduced to the case where $b = 1$, which is the base case for rank 2.

In rank three, the following equations reduce our general $\lambda = (a, b, c)$ for $c > 2$ to ψ functions evaluable by the base case. We again begin with the rank 2 Clebsch-Gordan result, this time lifted to rank 3.

$$\psi_{(a,b,c)} = \frac{\psi_{(a,b,2)}\psi_{(a,c,1)} - \psi_{(a,b,1)}\psi_{(a,c,2)}}{\psi_{(a,2,1)}}.$$

If $c = 2$ and $b > 3$, we use the Clebsch-Gordan in this way:

$$\psi_{(a,b,2)} = \psi_{(a,3,2)}\psi_{(b,2,1)} - \psi_{(a,2,1)}\psi_{(b,3,2)}.$$

If $c = 2$ and $b = 3$, we must instead employ Pieri's Rule:

$$\psi_{(a,3,2)} = \psi_{(a,3,1)}\psi_{(4,2,1)} - \psi_{(a+1,3,1)} - \psi_{(a,4,1)} - \psi_{(a,2,1)} - \psi_{(a-1,3,1)}.$$

Note that if $a = 4$, two of the partitions on the right hand side are not distinct, in this case those ψ functions are taken to be zero and the equality holds. So we have reduced to evaluating partitions with final entry 1. If $c = 1$ and $b > 3$, we can use Clebsch-Gordan to get:

$$\psi_{(a,b,1)} = \psi_{(a,3,1)}\psi_{(b,2,1)} - \psi_{(a,2,1)}\psi_{(b,3,1)}.$$

With $b = 3$, we use Pieri's Rule:

$$\psi_{(a,3,1)} = \psi_{(a,2,1)}\psi_{(4,2,1)} - \psi_{(a+1,2,1)} - \psi_{(a-1,2,1)},$$

and we have again managed to reduce to the base case, $c = 1$ and $b = 2$.

Look at the final two steps after we have reduced to the case of $c = 1$. We see that the method used is identical to the rank 2 reduction. This pattern remains; the rank r reduction is composed of a lifted rank 2 reduction and a rank $r - 1$ reduction, in the following way.

Given an arbitrary rank r, we begin with a partition $\lambda = (a_1, a_2, \ldots, a_r)$ and we assume $a_r > 2$. Then

$$\psi_\lambda = \frac{\psi_{(a_1,\dots,a_{r-2},a_{r-1},2)}\psi_{(a_1,\dots,a_{r-2},a_r,1)} - \psi_{(a_1,\dots,a_{r-2},a_{r-1},1)}\psi_{(a_1,\dots,a_{r-2},a_r,2)}}{\psi_{(a_1,\dots,a_{r-2},2,1)}}.$$

If $a_r = 2$, we instead need Pieri's Rule to get

$$\begin{aligned}\psi_{(a_1,\dots,a_{r-1},2)} = \psi_{(a_1,\dots,a_{r-1},1)}\psi_{(r+1,r-1,\dots,1)} \\ - \sum_{j=1}^{r-1}\left(\psi_{(a_1,\dots a_{j-1},a_j+1,a_{j+1}\dots,a_{r-1},1)} + \psi_{(a_1,\dots a_{j-1},a_j-1,a_{j+1}\dots,a_{r-1},1)}\right).\end{aligned}$$

Just as above, if any of the partitions in the subscripts in the sum are not distinct, that ψ function is taken to be zero. This reduces to rank r ψ functions with partitions that end in 1.

So then, given λ a length r partition ending in 1, we can create μ a length $r-1$ partition by subtracting 1 from each entry and dropping the final 0. The formulas used to reduce these rank $r-1$ ψ_μ functions all work for the ψ_λ functions with $\lambda_r = 1$. Thus we have inductively shown that for any rank r, the base case, Pieri's Rule, and Clebsch-Gordan results stated above give an algorithm for evaluating every $\psi_\lambda(\boldsymbol{x};\boldsymbol{t})$ uniquely.

5.5 A special value of ψ_λ

Now we will use a five-vertex model to prove the special value of Z_λ where the t_i's are set to 1.

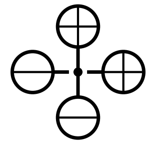
Proposition 5.14. *For λ a length r distinct partition and Z_λ computed with ψ-Boltzmann weights (see discussion concerning* (5.8)*), we have the following.*

$$Z_\lambda(\boldsymbol{x};1) = u' \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \prod_{i=1}^{r}(x_{\sigma(i)}^{\lambda_i-1} - x_{\sigma(i)}^{-\lambda_i}), \tag{5.30}$$

for u' the following power of $\sqrt{-1}$:

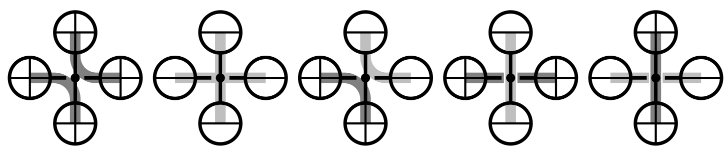
$$u' = \left(\sqrt{-1}\right)^{\frac{r(r-3)}{2}+\sum\lambda_i}. \tag{5.31}$$

Proof. Once we have set the $t_i = 1$, our six-vertex model is shrunk to a five-vertex model because the weight of the vertex filling



becomes zero (for both clear and black ice). The lack of this vertex filling makes the half-turn ice fillings much less flexible. This rigidity is easiest to see by altering the flowline viewpoint (first found in Remark 2.2) to best fit our argument. The flowlines below will only apply to arguments in this proof.

For the five vertex fillings in this specialized model, we use the flowlines in (5.32). We have only changed what happens when two negative flowlines meet at a vertex: now they are required to cross each other.



(5.32)

Under this new flowline scheme, negative flowlines cannot turn right. Since every flow enters from the top and exits at the right, each negative flowline must have a single left turn, and so it takes the shape of a large 'L'. The right boundary's U-shaped vertices each require exactly one of these negative flows. So the fillings may be listed by bijectively mapping the input flows $\lambda_1, \lambda_2, \ldots, \lambda_r$ to the r U-shaped vertices on the right boundary. For each mapping, there are 2^r choices of filling, since each flow can exit through the black or clear ice attached to its U-shaped vertex.

This gives a total of $2^r \cdot r!$ fillings, one for each element of the Weyl group. We first compute the evaluation of one filling: the one with no negative flowlines crossing, and all of them exiting out of the black ice. With careful calculating, this can be found to be

$$u' \prod_{i=1}^{r} x_i^{\lambda_i - 1},$$

for u' in (5.31).

Then, we find that if we switch the exit rows of two adjacent negative flowlines, causing them to cross, we simply negate the product and switch the powers of the two relevant $x_i's$. Finally, if we choose the clear ice exit for a given negative flowline which enters at λ_i, this negates the product and divides it by $x_i^{2\lambda_i - 1}$. Summing these evaluations into the equation for the partition function gives (5.30). ∎

With this formula, we may compute the special value of Z_ρ in the rank r case to be

$$\begin{aligned} Z_\rho(\boldsymbol{x};1) &= (\sqrt{-1})^{\frac{r(r-3)}{2}+\frac{r(r+1)}{2}} \sum_{\sigma\in S_r} \operatorname{sgn}(\sigma) \prod_{i=1}^{r} (x_{\sigma(i)}^{(r+1-i)-1} - x_{\sigma(i)}^{-(r+1-i)}) \\ &= (-1)^{\frac{r(r-1)}{2}} \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \prod_{i=1}^{r} (x_{\sigma(i)}^{(r+1-i)-1} - x_{\sigma(i)}^{-(r+1-i)}). \end{aligned}$$

Thus we obtain the following formula for the special value $\psi_\lambda(\boldsymbol{x};1)$:

$$\psi_\lambda = \frac{Z_\lambda}{uZ_\rho} = \frac{u' \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \prod_{i=1}^{r} (x_{\sigma(i)}^{\lambda_i-1} - x_{\sigma(i)}^{-\lambda_i})}{u(-1)^{\frac{r(r-1)}{2}} \sum_{\sigma\in S_r} \operatorname{sgn}\sigma \prod_{i=1}^{r} (x_{\sigma(i)}^{(r+1-i)-1} - x_{\sigma(i)}^{-(r+1-i)})} = \operatorname{char}(V_\mu) \tag{5.33}$$

(a simple calculation shows $u' \cdot u^{-1} \cdot (-1)^{\frac{r(r-1)}{2}} = 1$, for u defined in Remark 5.3). The final equality of (5.33) is a restatement of the Weyl character formula in type B for V_μ the highest weight representation of $\mathrm{SO}(2r+1)$ with highest weight the (not necessarily distinct) partition μ, where $\mu + \rho = \lambda$. So we have the special value of $\psi_{\mu+\rho}(\boldsymbol{x};1)$ expected from the conjectured recursion.

Theorem 5.15 (Special value of ψ_λ)**.** *For μ a length r partition, $\rho = (r, r-1, \ldots, 1)$, and V_μ the unique representation of $\mathrm{SO}_{2r+1}(\mathbb{C})$ of highest weight μ,*

$$\psi_{\mu+\rho}(\boldsymbol{x};1) = \operatorname{char}(V_\mu).$$

Chapter 6

Whittaker Coefficients of an Sp_4 Eisenstein Series

In this chapter we describe a calculation of the Whittaker coefficients of a minimal parabolic Sp_4 Eisenstein series. Our approach parallels the work done by Brubaker, Bump, and Friedberg [4] on SL_{r+1} Eisenstein series. The goal is to write these Whittaker coefficients in terms of the Whittaker coefficients of an Sp_2 $(= \mathrm{SL}_2)$ Eisenstein series. We then discuss methods to generalize the work to higher rank groups of Cartan type C (Sp_{2r} for $r > 2$), which would enable a recursive expression of the Whittaker coefficients of Sp_{2r} Eisenstein series.

We obtain this expression by an explicit decomposition of the unipotent radical of our Borel subgroup according to a convenient ordering of the positive roots. In particular, the ordering behaves nicely with respect to induction on rank. Our results may be viewed as an alternate approach to the Casselman-Shalika formula for Sp_{2r} Eisenstein series. The idea behind this uniporent factorization is implicit in [4], but is explicitly described in [**?**] in this context and [**?**] in the context of p-adic Whittaker functions. However, each special case presents its own intricate details, and our main advance here is to carry these out for the symplectic group.

Furthermore, this approach is amenable to generalization on the metaplectic group as defined by Matsumoto [**?**] and Kubota [**?**]. These calculations on covering groups require many evaluations of the cocycle of the group. The block-compatability of this cocycle, proven by Banks, Levi, and Sepanski [1], implies that our decomposition of the unipotent radical reduces these evaluations to the Kubota symbol on SL_2.

This computation is independent of the work on half-turn ice from the earlier chapters of the thesis. However, there is hope that the two calculations will eventually find common ground. In Theorems 3.10 and 5.15, we showed that the partition function of half-turn ice factors into a deformation of the Weyl denominator and a deformation of a highest weight character. We suspect that this deformed character could be a value of the spherical Whittaker function on some algebraic group related to SO_{2r+1}, by a metaplectic extension of the Casselman-Shalika formula.

Deciding this question remains out of reach to date. That being said, if the generalizations of the work in this chapter to higher rank and metaplectic covers are realized, we will have a recursive definition for the Whittaker coefficients of these Eisenstein series. This recursion may be comparable to the algorithm developed for our half-turn ice partition function, either generalizing a version of type B or type C denominators.

6.1 Preliminary Definitions

Our setup will follow that of Brubaker, Bump, and Friedberg [4], and other papers about similar computations (e.g. [**?**] and [**?**]).

Let F be a number field with ring of integers $\mathfrak{O}$. We choose a finite set S of places of F such that the following conditions hold.

1. S contains all the archimedean places of F.
2. S contains all the places of F ramified over .
3. The subring of F consisting of x with $|x|_v \leq 1$ for $v \notin S$ is a principal ideal domain. We call this subring $\mathfrak{O}_S$.

Such an S may be chosen by inverting sufficiently many places corresponding to representatives in the ideal class group.

We define the ring $F_S = \prod_{v \in S} F_v$. Both F and $\mathfrak{O}_S$ may be embedded diagonally into F_S, and this embedding of $\mathfrak{O}_S$ is discrete and cocompact.

For any ring R, we define $\mathrm{Sp}_4(R)$ to be the set of 4×4 matrices g over R such that

$$ {}^t g\, w_0 g w_0^{-1} = I. $$

Here

$$ w_0 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}, \qquad I = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, $$

and ${}^t g$ denotes the transpose of g.

We define the representation $\pi(s_1, s_2)$ of $\mathrm{Sp}_4(F_S)$ as the space of smooth functions $f : \mathrm{Sp}_4(F_S) \longrightarrow \mathbb{C}$ satisfying

$$f\left(\begin{pmatrix} t_2 & * & * & * \\ & t_1 & * & * \\ & & t_1^{-1} & * \\ & & & t_2^{-1} \end{pmatrix} g\right) = |t_1|^{2s_1}|t_2|^{2s_1+2s_2} f(g) \tag{6.1}$$

with the action of right translation. Then for any $f \in \pi(s_1, s_2)$ we define the Eisenstein series $E_f : \mathrm{Sp}_4(F_S) \longrightarrow \mathbb{C}$ by

$$E_f(g) = \sum_{\gamma} f(\gamma g), \tag{6.2}$$

where γ is summed over right coset representatives of the quotient

$$B_{\mathrm{Sp}(4)}(\mathfrak{O}_S) \backslash \mathrm{Sp}_4(\mathfrak{O}_S)$$

for $B_{\mathrm{Sp}(4)}$ the Borel subgroup of upper triangular matrices of Sp_4.

We will study this Eisenstein series by computing its Whittaker coefficients. With an eye toward generalizing to arbitrary rank, we will show that the computation reduces to one over $\mathrm{Sp}_2(F_S) = \mathrm{SL}_2(F_S)$. This reduction is possible for the larger even symplectic groups, and so the computation generalizes to a recursive calculation of the Whittaker coefficients for $\mathrm{Sp}_{2r}(F_S)$ in terms of those for $\mathrm{Sp}_{2(r-1)}(F_S)$.

We define four embeddings $\iota_* : \mathrm{SL}_2(F_S) \longrightarrow \mathrm{Sp}_4(F_S)$ as follows:

$$\iota_1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & & b \\ & 1 & & \\ & & 1 & \\ c & & & d \end{pmatrix} \qquad \iota_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & b & \\ & a & & b \\ c & & d & \\ & c & & d \end{pmatrix}$$

$$\iota_3\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b & & \\ -c & d & & \\ & & a & b \\ & & c & d \end{pmatrix} \qquad \iota'\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & a & b & \\ & c & d & \\ & & & 1 \end{pmatrix}$$

We also define the function

$$\Theta_f(g) = \sum_{\gamma'} f\left(\iota'(\gamma')g\right), \tag{6.3}$$

where γ' is summed over $B_{\mathrm{SL}(2)}(\mathfrak{O}_S)\backslash\,\mathrm{SL}_2(\mathfrak{O}_S)$ for $B_{\mathrm{SL}(2)}$ the Borel subgroup of upper triangular matrices of SL_2 .

We prove a useful reformulation of E_f in terms of Θ.

Proposition 6.1.

$$E_f(g) = \sum_{\gamma \in P(\mathfrak{O}_S)\backslash\,\mathrm{Sp}_4(\mathfrak{O}_S)} \Theta_f(\gamma g), \tag{6.4}$$

where P is the parabolic subgroup of Sp_4 of the form

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & * & * & * \\ & & & * \end{pmatrix} \in \mathrm{Sp}_4 \right\}.$$

Proof. The equality (6.4) can be found by expanding its RHS using (6.3) into the double sum

$$\sum_{\gamma} \sum_{\gamma'} f\left(\iota'(\gamma')\gamma g\right).$$

Now assume that two choices of γ_i and γ_i' $(i = 1, 2)$ in the indices of this double sum are redundant, i.e., the two products $\iota'(\gamma_i')\gamma_i$ lie in the same right coset of $B_{\mathrm{Sp}(4)}$. This is equivalent to

$$\iota'(\gamma_1')\gamma_1\gamma_2^{-1}\iota'(\gamma_2')^{-1} \in B_{\mathrm{Sp}(4)}(\mathfrak{O}_S) \subset P(\mathfrak{O}_S).$$

From this we have $\gamma_1\gamma_2^{-1} \in P(\mathfrak{O}_S)$, so these two matrices represent the same coset, and thus are equal. Since they cancel, we have that $\gamma_1'\gamma_2'^{-1} \in B_{\mathrm{SL}(2)}(\mathfrak{O}_S)$. So the double sum gives a bijection with a set of coset representatives for $B_{\mathrm{Sp}(4)}(\mathfrak{O}_S)\backslash\,\mathrm{Sp}_4(\mathfrak{O}_S)$. ■

6.2 The Whittaker coefficients

Let $N_{\mathrm{Sp}(4)}$ be the upper triangular unipotent subgroup of Sp_4, so for any ring R,

$$N_{\mathrm{Sp}(4)}(R) = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \in \mathrm{Sp}_4(R) \right\}.$$

Then for χ a character of $N_{\mathrm{Sp}(4)}(F_S)$ trivial on $N_{\mathrm{Sp}(4)}(\mathfrak{O}_S)$, the Whittaker coefficient of E_f indexed by χ is

$$\int_{N_{\mathrm{Sp}(4)}(\mathfrak{O}_S)\backslash N_{\mathrm{Sp}(4)}(F_S)} E_f(w_0 n)\bar{\chi}(n)dn. \tag{6.5}$$

In this chapter our aim is to compute (6.5) explicitly, so we will use the particular structure of Sp_4. A short calculation shows that the general form of $n \in N_{\mathrm{Sp}(4)}(R)$ is

$$\begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ & 1 & x_{23} & x_{13} - x_{12}x_{23} \\ & & 1 & -x_{12} \\ & & & 1 \end{pmatrix}, \tag{6.6}$$

for $x_{ij} \in R$. We use the shorthand $n_+(x_{ij}) = n_+(x_{12}, x_{13}, x_{14}, x_{23})$ to notate the matrix (6.6). We may then use the x_{ij} as our variables of integration. The region of integration and differential become

$$N_{\mathrm{Sp}(4)}(\mathfrak{O}_S)\backslash N_{\mathrm{Sp}(4)}(F_S) \mapsto (F_S/\mathfrak{O}_S)^4$$

and

$$dn \mapsto dx_{12}dx_{13}dx_{14}dx_{23} = \prod dx_{ij}.$$

We will write $\mathcal{R}$ for the compact region $F_S/\mathfrak{O}_S$ to tidy up our integrals.

Fix an additive character $\psi : F_S \longrightarrow \mathbb{C}$ with conductor $\mathfrak{O}_S$. Then there is a choice of $m_1, m_2 \in \mathfrak{O}_S$ that gives the following reformulation of the character χ:

$$\chi(n_+(x_{ij})) = \psi(m_1 x_{12} + m_2 x_{23}). \tag{6.7}$$

Thus our Whittaker coefficients are indexed by pairs $(m_1, m_2) \in \mathfrak{O}_S^2$.

Combining the above observations, we may rewrite (6.5) as

$$\int_{\mathcal{R}^4} E_f \left(w_0 \begin{pmatrix} 1 & x_{12} & x_{13} & x_{14} \\ & 1 & x_{23} & x_{13} - x_{12}x_{23} \\ & & 1 & -x_{12} \\ & & & 1 \end{pmatrix} \right) \bar{\psi}(m_1 x_{12} + m_2 x_{23}) \prod dx_{ij}. \tag{6.8}$$

Substituting (6.4) into (6.8) gives

$$\int_{\mathcal{R}^4} \sum_{\gamma \in P(\mathfrak{O}_S)\backslash \mathrm{Sp}_4(\mathfrak{O}_S)} \Theta_f(\gamma w_0 n_+(x_{ij}))\psi(-m_1 x_{12} - m_2 x_{23}) \prod dx_{ij}. \tag{6.9}$$

We will determine a parametrization of γ in terms of its bottom row, whose entries we label (B_4, B_3, B_2, B_1) from left to right. First, we show that we may ignore those γ with $B_1 = 0$ with the following lemma.

Lemma 6.2. *The summands in* (6.9) *that have* γ *with bottom right entry zero have trivial integral.*

Proof. Consider the integral of the summand for a fixed γ with bottom row $(B_4, B_3, B_2, 0)$. Using (6.3),

$$\int_{\mathcal{R}^4} \Theta_f \left(\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ B_4 & B_3 & B_2 & 0 \end{pmatrix} w_0 n_+(x_{ij}) \right) \psi(-m_1 x_{12} - m_2 x_{23}) \prod dx_{ij}$$

$$= \sum_{\gamma'} \int_{\mathcal{R}^4} f \left(\iota'(\gamma') \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & -B_2 & B_3 & B_4 \end{pmatrix} n_+(x_{ij}) \right) \psi(-m_1 x_{12} - m_2 x_{23}) \prod dx_{ij}.$$

So for each fixed $\gamma' \in P(\mathfrak{O}_S)\backslash \mathrm{Sp}_4(\mathfrak{O}_S)$, we observe that the product of the matrices

$$\iota'(\gamma') w_0 \gamma = \begin{pmatrix} 1 & & \\ & \begin{pmatrix} \gamma' \end{pmatrix} & \\ & & 1 \end{pmatrix} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & -B_2 & B_3 & B_4 \end{pmatrix}$$

has bottom left entry is zero.

This zero entry implies that in the Bruhat decomposition, $\iota'(\gamma')w_0\gamma \in B_{\mathrm{Sp}(4)}wB_{\mathrm{Sp}(4)}$ for some $w \neq w_0$ in the Weyl group W of Sp_4. Since $w \neq w_0$, there is a simple positive root α such that $w(\alpha)$ is still positive. There are two simple roots in Sp_4, so we have two cases.

If the long root, α_1, has $w(\alpha_1) > 0$, we may write that

$$w \begin{pmatrix} 1 & & & \\ & 1 & x_{23} & \\ & & 1 & \\ & & & 1 \end{pmatrix} w^{-1} \in B_{\mathrm{Sp}(4)}(F)$$

for any $x_{23} \in F$. In the second case, $w(\alpha_2) > 0$ for the short root α_2. Here we have

$$w \begin{pmatrix} 1 & x_{12} & & \\ & 1 & & \\ & & 1 & -x_{12} \\ & & & 1 \end{pmatrix} w^{-1} \in B_{\mathrm{Sp}(4)}(F).$$

Whichever case, we can factor $n_+(x_{ij})$ and bring the appropriate matrix across w, keeping it in $N_{\mathrm{Sp}(4)}(F_S)$. Then using the left-translation property of f, we see that the integral over either x_{23} (case 1) or x_{12} (case 2) will be trivial, because the value of f is independent of the variable of integration and ψ integrates to 0. ∎

So we remove from the sum all terms with bottom right entry of γ zero.

Multiplying γ by some $p \in P(\mathfrak{O}_S)$ on the left, we see that the bottom row of γ is simply multiplied by the bottom right entry of p, some element of $\mathfrak{O}_S^\times$. So we may parametrize the remaining summands by the 4-tuples (B_4, B_3, B_2, B_1) with

- $B_1 \in (\mathfrak{O}_S \setminus \{0\})/\mathfrak{O}_S^\times$,
- $B_i \in \mathfrak{O}_S$ for $i = 2, 3, 4$, and
- $\gcd(B_1, B_2, B_3, B_4) = 1$.

That each 4-tuple is represented as the bottom row of some $\gamma \in \mathrm{Sp}_4(\mathfrak{O}_S)$ is a consequence of the following lemma.

Lemma 6.3. *Given (B_4, B_3, B_2, B_1) satisfying the three conditions above, there are* *unique $c_1, c_3 \in \mathfrak{O}_S$ and $d_1, d_2, d_3 \in (\mathfrak{O}_S \setminus \{0\})/\mathfrak{O}_S^\times$ such that the product*

$$\iota_2 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \iota_1 \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \iota_3 \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \tag{6.10}$$

has bottom row (B_4, B_3, B_2, B_1) for a proper choice of the other matrix entries (of course, these choices must also place the argument of each ι_i in $\mathrm{SL}_2(\mathfrak{O}_S)$). Furthermore, a choice of $a_3, b_3 \in \mathfrak{O}_S$ integers satisfying $a_3d_3 - b_3c_3 = 1$ fixes $c_2 \in \mathfrak{O}_S$ uniquely.

Proof. We begin with the 4-tuple (B_4, B_3, B_2, B_1) and, assuming that such a factorization exists, we prove the declared uniqueness of the matrix entries. Multiplying out the product (6.3), we get

$$\begin{pmatrix} a_2 & & b_2 & \\ & a_2 & & b_2 \\ c_2 & & d_2 & \\ & c_2 & & d_2 \end{pmatrix} \begin{pmatrix} a_1 & & & b_1 \\ & 1 & & \\ & & 1 & \\ c_1 & & & d_1 \end{pmatrix} \begin{pmatrix} a_3 & -b_3 & & \\ -c_3 & d_3 & & \\ & & a_3 & b_3 \\ & & c_3 & d_3 \end{pmatrix}$$
$$= \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ a_3c_1d_2 - c_2c_3 & c_2d_3 - b_3c_1d_2 & c_3d_1d_2 & d_1d_2d_3 \end{pmatrix}. \tag{6.11}$$

From (6.11), we have that $d_1d_2d_3 = B_1$ and $c_3d_1d_2 = B_2$. Since we have $\gcd(c_i, d_i) = 1$ for each of the assumed factors, we determine that

$$d_1d_2 = \gcd(B_1, B_2), \qquad d_3 = \frac{B_1}{\gcd(B_1, B_2)}, \qquad \text{and} \qquad c_3 = \frac{B_2}{\gcd(B_1, B_2)}$$

(we remind the reader that $\gcd(a, 0) = a$ for any a).

Now that we have fixed c_3 and d_3, we compute that

$$d_3B_4 + c_3B_3 = d_3(a_3c_1d_2 - c_2c_3) + c_3(c_2d_3 - b_3c_1d_2) = (a_3d_3 - b_3c_3)c_1d_2 = c_1d_2.$$

So having fixed c_1d_2 and d_1d_2, we use the fact that $\gcd(c_1, d_1) = 1$ to compute the following three matrix entries.

$$d_2 = \gcd(d_1d_2, c_1d_2) \qquad d_1 = \frac{d_1d_2}{d_2} \qquad c_1 = \frac{c_1d_2}{d_2}$$

This shows the claimed uniqueness of all the entries but c_2. Now, we can choose $a_3, b_3 \in \mathfrak{O}_S$ such that $a_3d_3 - b_3c_3 = 1$. Then we use these values to determine c_2 uniquely:

$$b_3B_4 + a_3B_3 = b_3(a_3c_1d_2 - c_2c_3) + a_3(c_2d_3 - b_3c_1d_2) = (a_3d_3 - b_3c_3)c_2 = c_2.$$

We have determined the values of each c_i and d_i, and so we can choose a_i and b_i for $i = 1, 2$ so that $a_id_i - b_ic_i = 1$. Then we have found formulae for the matrix entries in the above factorization, and we have also demonstrated that every bottom row in our parametrization has a representative in $\mathrm{Sp}_4(\mathfrak{O}_S)$. ■

We will let $\gamma(B_4, B_3, B_2, B_1) \in \mathrm{Sp}_4(\mathfrak{O}_S)$ be the product of these factors, determined in the above way from the values B_i.

Remark 6.4. This particular factorization is chosen because the relationship between the B_i and the c_i and d_i is as simple as possible. However, the order of matrices in (6.10) could be predicted by studying the root system of Sp_4. We choose a minimal decomposition (into simple reflections) of the long element of the Weyl group. The two simple roots correspond to the reflections below.

$$\sigma_1 = \begin{pmatrix} 1 & & & \\ & & -1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} & 1 & & \\ -1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$$

One choice of decomposition is $\sigma_2\sigma_1\sigma_2\sigma_1 = w_0$. This decomposition gives an ordering on the positive roots of Sp_4 as follows (see for example, Proposition 21.10 of Bump [?]). For Φ_+ the set of positive roots of Sp_4 and $w \in W$, we defineThen consider the sets ${}^{w}\Phi_+$ for each of the partial products w in our choice of decomposition of w_0:

$$\begin{aligned} {}^{\sigma_1}\Phi_+ &= \{\alpha_1\}, \\ {}^{\sigma_2\sigma_1}\Phi_+ &= \{\alpha_1, \alpha_1 + \alpha_2\}, \\ {}^{\sigma_1\sigma_2\sigma_1}\Phi_+ &= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}, \\ {}^{w_0}\Phi_+ &= \Phi_+. \end{aligned}$$

This gives rise to an ordering of roots: $\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2$. The ordering suggests the ordering of matrices as in (6.10), because each positive root corresponds to an embedding of SL_2.

$$\begin{aligned}\alpha_1 + 2\alpha_2 &\leftrightarrow \iota_1\\ \alpha_1 + \alpha_2 &\leftrightarrow \iota_2\\ \alpha_2 &\leftrightarrow \iota_3\\ \alpha_1 &\leftrightarrow \iota'\end{aligned}$$

Thus, we would expect the ordering $\iota_2, \iota_1, \iota_3$ when inducing from $\iota'(\mathrm{SL}_2)$ to Sp_4.

Using Lemma 6.3 we rewrite the Whittaker coefficient as

$$\int\limits_{\mathcal{R}^4} \sideset{}{'}\sum_{\substack{B_1 \in (\mathfrak{O}_S \setminus \{0\})/\mathfrak{O}_S^\times \\ B_2, B_3, B_4 \in \mathfrak{O}_S}} \Theta_f(\gamma(B_i) w_0 n_+(x_{ij})) \psi(-m_1 x_{12} - m_2 x_{23}) \prod dx_{ij}.$$

The sum is primed because we require $\gcd(B_1, B_2, B_3, B_4) = 1$. Next we 'unfold' the integral using the following proposition.

Proposition 6.5. *For a fixed nonzero $B_1 \in \mathfrak{O}_S$, we have the equality*

$$\begin{aligned}&\int\limits_{\mathcal{R}^4} \sideset{}{'}\sum_{B_2, B_3, B_4 \in \mathfrak{O}_S} \Theta_f(\gamma(B_i) w_0 n_+(x_{ij})) \psi(-m_1 x_{12} - m_2 x_{23}) dx_{12} dx_{13} dx_{14} dx_{23}\\ &\quad = \int\limits_{\mathcal{R}} \left[\int_{F_S^3} \sideset{}{'}\sum_{B_2, B_3, B_4 \ (B_1)} \Theta_f(\gamma(B_i) w_0 n_+(x_{ij})) \psi(-m_1 x_{12} - m_2 x_{23}) \prod_{j=2}^{4} dx_{1j} \right] dx_{23}.\end{aligned}$$

Proof. We observe the effect of an insertion of a matrix of the form

$$n_+(a, b, c, 0) = \begin{pmatrix} 1 & a & b & c \\ & 1 & & b \\ & & 1 & -a \\ & & & 1 \end{pmatrix} \qquad \text{for } a, b, c \in \mathfrak{O}_S$$

between w_0 and $n_+(x_{ij})$ in the argument of Θ_f. If we combine this matrix with $n_+(x_{ij})$, we find that

$$\begin{aligned}&n_+(a, b, c, 0) n_+(x_{12}, x_{13}, x_{14}, x_{23})\\ &\qquad = n_+(x_{12} + a, x_{13} + a x_{23} + b, x_{14} + a(x_{13} - x_{12} x_{23}) - b x_{12} + c, x_{23}).\end{aligned}$$

We observe that letting a, b, c vary over $\mathfrak{O}_S$ and summing is equivalent to expanding the integrals of x_{12}, x_{13}, x_{14} from $F_S/\mathfrak{O}_S$ to all of F_S. The easiest way to see this is first to fix a and b and sum varying c, which expands x_{14}. Then if we just fix a and sum over b, we are expanding x_{13}. Finally, summing over a expands x_{12}.

Now we instead move the matrix $n_+(a, b, c, 0)$ across w_0 and observe its effect on $\gamma(B_i)$. The computation is

$$\gamma(B_i)w_0 n_+(a,b,c,0)w_0^{-1} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ B_4 - aB_3 + bB_2 + cB_1 & B_3 + bB_1 & B_2 + aB_1 & B_1 \end{pmatrix}.$$

Here, if we sum a, b, c over all of $\mathfrak{O}_S$ it is equivalent to summing B_2, B_3, and B_4 over a fixed set of residues of $\mathfrak{O}_S$ mod B_1. In other words, for a fixed B_1, if we let B_2, B_3, and B_4 vary over a fixed set of representatives mod B_1 and let a, b, and c vary over all $\mathfrak{O}_S$, that is equivalent to B_2, B_3, and B_4 varying over all of $\mathfrak{O}_S$.

These arguments show that we can insert $n_+(a, b, c, 0)$ into the integral, with a, b, c summed over $\mathfrak{O}_S$, and reduce the sum over B_i to be summed mod B_1 for $i > 1$. Then we can move the inserted matrix to $n_+(x_{ij})$ and remove it by expanding the integrals of x_{1j}. So in short, for fixed B_1,

$$\sum_{B_2,B_3,B_4 \in \mathfrak{O}_S} \int_{\mathcal{R}^4} * \prod dx_{ij} = \sum_{B_2,B_3,B_4 \ (B_1)} \int_{\mathcal{R}} \left[\int_{F_S^3} * \prod_{j=2}^{4} dx_{1j} \right] dx_{23}. \qquad \blacksquare$$

This gives the following form of the Whittaker coefficient:

$$\int_{\mathcal{R}} \int_{F_S^3} \sideset{}{'}\sum_{\substack{B_1 \in (\mathfrak{O}_S \setminus \{0\})/\mathfrak{O}_S^\times \\ B_2,B_3,B_4 \mod B_1}} \Theta_f(\gamma(B_i)w_0 n_+(x_{ij}))\psi(-m_1 x_{12} - m_2 x_{23}) \prod_{j=2}^{4} dx_{1j} dx_{23}. \tag{6.12}$$

Now we want to substitute the factorization of $\gamma(B_i)$ from Lemma 6.3 and convert this parametrization over B_i to one over c_i and d_i.

From the proof of Lemma 6.3, we see that given a set of B_i, we can directly compute the c_i and d_i, except that c_2 can run over a set of elements of a single residue mod $c_1 d_2$, which is fixed only when a choice of a_3 and b_3 is made. Now we ask: if we only vary B_i, $i > 1$, over a fixed set of residues mod B_1, what do the c_i and d_i vary over?

Lemma 6.6. *For a fixed nonzero $B_1 \in \mathfrak{O}_S$, define the set*

$$D_{B_1} = \{(d_1, d_2, d_3) \in ((\mathfrak{O}_S \setminus \{0\})/\mathfrak{O}_S^\times)^3 | d_1 d_2 d_3 = B_1\}.$$

For a fixed $(d_1, d_2, d_3) \in D_{B_1}$, let C_{d_1,d_2,d_3} be a set of triples (c_1, c_2, c_3) such that each c_i runs over a fixed set of representatives of the following moduli:

$$c_1 \pmod{d_1 d_3^2}, \qquad c_2 \pmod{d_1 d_2}, \qquad \text{and} \qquad c_3 \pmod{d_3},$$

that satisfy $(c_i, d_i) = 1$ for $i = 1, 2, 3$. Then there is a bijection between the parameters B_i, $i > 1$, in the sum in (6.12) under B_1 and the set

$$\{((d_1, d_2, d_3), (c_1, c_2, c_3)) \in D_{B_1} \times \mathfrak{O}_S^3 | (c_1, c_2, c_3) \in C_{d_1,d_2,d_3}\}$$

that respects the factorization in Lemma 6.3.

Proof. We will show the equivalence of these indices by the following procedure. For each B_1, we choose a fixed complete set of representatives of $\mathfrak{O}_S/(B_1\mathfrak{O}_S)$, M_{B_1}. Now, given any 4-tuple (B_4, B_3, B_2, B_1), with B_2, B_3, B_4 ranging over all of $\mathfrak{O}_S$, we can find c_i and d_i as described above. Now we perform the following substitutions to move (B_4, B_3, B_2, B_1) to $M_{B_1}^3 \times \{B_1\}$, so each B_i with $i > 1$ is one of the chosen set of representatives.

First, we choose $t_2 \in \mathfrak{O}_S$ so that $B_2 + t_2 B_1 \in M_{B_1}$ and perform the following replacements.

$$\begin{aligned} B_2 &\longmapsto B_2 + t_2 B_1 \\ B_4 &\longmapsto B_4 - t_2 B_3 \\ c_3 &\longmapsto c_3 + t_2 d_3 \\ a_3 &\longmapsto a_3 + t_2 b_3 \end{aligned}$$

Next, we choose t_3 such that $B_3 + t_3 B_1 \in M_{B_1}$ and substitute the following.

$$\begin{aligned} B_3 &\longrightarrow B_3 + t_3 B_1 \\ B_4 &\longrightarrow B_4 - t_3 d_1 d_2 c_3 \\ c_2 &\longrightarrow c_2 + t_3 d_1 d_2 \end{aligned}$$

Finally, we correct B_4 with a proper t_4.

$$\begin{aligned} B_4 &\longrightarrow B_4 + t_4 B_1 \\ c_1 &\longrightarrow c_1 + t_4 d_1 d_3^2 \\ c_2 &\longrightarrow c_2 + t_4 b_3 d_1 d_2 d_3 \end{aligned}$$

The important properties of these subsequent substitutions are that the output c_i are equivalent to the input c_i modulo the following.

$$\begin{aligned} &c_1 \pmod{d_1 d_3^2} \\ &c_2 \pmod{d_1 d_2} \\ &c_3 \pmod{d_3} \end{aligned} \tag{6.13}$$

Also, crucially, in the t_3 substitution, B_2 is not changed, so it stays in M_{B_1}. Similarly, in the final substitution, B_2 and B_3 stay fixed, so the output B_i are in M_{B_1} for $i = 2, 3, 4$. This implies that any chosen cosets c_1, c_2, c_3 modulo $d_1 d_3^2, d_1 d_2, d_3$ respectively, with $\gcd(c_i, d_i) = 1$, are represented by some 4-tuple in $\{B_1\} \times M_{B_1}^3$. To show the equivalence, we want this surjection to be a bijection. For each B_1, these two sets of cosets are finite. Since we already have the surjection, that is, that every valid choice of c_i is represented, we need only show that the size of these sets is equal.

For a given B_1, we want to count the number of triples $(B_2, B_3, B_4) \in M_{B_1}$ with $\gcd(B_1, B_2, B_3, B_4) = 1$. Let $D_1 = \gcd(B_1, B_2, B_3)$, $D_1 D_2 = \gcd(B_1, B_2)$, and $D_1 D_2 D_3 = B_1$. If we fix these D_i, then B_4 is chosen from $\varphi(D_1) D_2 D_3$ possible cosets mod B_1, since it must be relatively prime to D_1. Also, B_3 is divisible by D_1 and so the quotient runs over cosets mod $D_2 D_3$ that are relatively prime to D_2. The number of cosets is $\varphi(D_2) D_3$. Finally, B_2 is divisible by $D_1 D_2$, and the quotient runs over $\varphi(D_3)$ cosets. Thus the total number of choices of (B_2, B_3, B_4) for a fixed B_1 and D_i is

$$\varphi(D_1)\varphi(D_2)\varphi(D_3) D_2 D_3^2.$$

Since we would like the D_i to vary, the total number of valid 4-tuples with a given B_1 is

$$\sum_{D_1 D_2 D_3 = B_1} \varphi(D_1)\varphi(D_2)\varphi(D_3) D_2 D_3^2.$$

Here the sum is taken so that the D_i are chosen modulo $\mathfrak{O}^\times$, since they only represent the greatest common divisor of integers.

Now to count the other parametrization. Given a B_1, the d_i must multiply to B_1.

Then, given a choice of d_i, the c_i run over a set of cosets according to (6.13). The numbers of choices are $\varphi(d_1)d_3^2$ for c_1, $\varphi(d_2)d_1$ for c_2, and $\varphi(d_3)$ for c_3. Thus the total number of choices of d_i and c_i for a fixed B_1 is

$$\sum_{d_1d_2d_3=B_1} \varphi(d_1)d_3^2\varphi(d_2)d_1\varphi(d_3),$$

the exact same formula for the number of choices of the other set of parameters. This shows that the map between the two parametrizations is a bijection, and thus they are equivalent. ■

Before we get back to the Whittaker coefficient, let us study this factorization of $\gamma(B_i)$ further. We have factored it into the three matrices

$$\begin{pmatrix} a_2 & & b_2 & \\ & a_2 & & b_2 \\ c_2 & & d_2 & \\ & c_2 & & d_2 \end{pmatrix} \begin{pmatrix} a_1 & & & b_1 \\ & 1 & & \\ & & 1 & \\ c_1 & & & d_1 \end{pmatrix} \begin{pmatrix} a_3 & -b_3 & & \\ -c_3 & d_3 & & \\ & & a_3 & b_3 \\ & & c_3 & d_3 \end{pmatrix}. \tag{6.14}$$

Each of these matrices is an element of an embedding $\iota_i \operatorname{SL}_2(F_S) \longrightarrow \operatorname{Sp}_4(F_S)$. Since each $d_i \in \mathfrak{O}_S$ and is nonzero, we can use the following $\operatorname{SL}_2(F_S)$ decomposition for each factor:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \frac{b}{d} \\ & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{c}{d} & 1 \end{pmatrix}, \tag{6.15}$$

which leaves us with a total of nine matrices in the factorization. Using this decomposition, we can rearrange (6.14) to get the following result.

Lemma 6.7. *The factorization of $\gamma(B_i)$ in* (6.14) *can be rewritten as*

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \mathfrak{D}_{d_1,d_2,d_3} \begin{pmatrix} 1 & & & \\ -\frac{c_3}{d_3} & 1 & & \\ * & -\frac{2b_3c_2d_3}{d_1d_2} + \frac{b_3^2c_1}{d_1} & 1 & \\ * & * & \frac{c_3}{d_3} & 1 \end{pmatrix} \tag{6.16}$$

for $\mathfrak{D}_{d_1,d_2,d_3}$ the diagonal matrix

$$\begin{pmatrix} d_1^{-1}d_2^{-1}d_3^{-1} & & & \\ & \frac{d_3}{d_2} & & \\ & & \frac{d_2}{d_3} & \\ & & & d_1d_2d_3 \end{pmatrix}.$$

Proof. We first pull the three upper triangular matrices of the nine-matrix decomposition to the left of the entire product. On the far left we already have

$$\begin{pmatrix} 1 & & \frac{b_2}{d_2} & \\ & 1 & & \frac{b_2}{d_2} \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Next, to move the matrix below (the upper triangular matrix from the ι_1 embedding) across the two matrices to its left (the ones above the arrows), we conjugate it by these two matrices in sequence. So the arrow represents conjugation by the matrix above the arrow.

$$\begin{pmatrix} 1 & & & \frac{b_1}{d_1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \xrightarrow{\iota_2\begin{pmatrix} 1 & \\ c_2/d_2 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{b_1c_2}{d_1d_2} & & \frac{b_1}{d_1} \\ & 1 & & \\ & & 1 & \frac{b_1c_2}{d_1d_2} \\ & & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{b_1c_2}{d_1d_2} & & \frac{b_1}{d_1} \\ & 1 & & \\ & & 1 & \frac{b_1c_2}{d_1d_2} \\ & & & 1 \end{pmatrix} \xrightarrow{\iota_2\begin{pmatrix} d_2^{-1} & \\ & d_2 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{b_1c_2}{d_1d_2} & & \frac{b_1}{d_1d_2^2} \\ & 1 & & \\ & & 1 & \frac{b_1c_2}{d_1d_2} \\ & & & 1 \end{pmatrix}$$

The rightmost upper triangular unipotent matrix is conjugated by four matrices.

$$\begin{pmatrix} 1 & -\frac{b_3}{d_3} & & \\ & 1 & & \\ & & 1 & \frac{b_3}{d_3} \\ & & & 1 \end{pmatrix} \xrightarrow{\iota_1\begin{pmatrix} 1 & \\ c_1/d_1 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{b_3}{d_3} & & \\ & 1 & & \\ -\frac{b_3c_1}{d_1d_3} & & 1 & \frac{b_3}{d_3} \\ & -\frac{b_3c_1}{d_1d_3} & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{b_3}{d_3} & & \\ & 1 & & \\ -\frac{b_3c_1}{d_1d_3} & & 1 & \frac{b_3}{d_3} \\ & -\frac{b_3c_1}{d_1d_3} & & 1 \end{pmatrix} \xrightarrow{\iota_1\begin{pmatrix} d_1^{-1} & \\ & d_1 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ -\frac{b_3c_1}{d_3} & & 1 & \frac{b_3}{d_1d_3} \\ & -\frac{b_3c_1}{d_3} & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ -\frac{b_3c_1}{d_3} & & 1 & \frac{b_3}{d_1d_3} \\ & -\frac{b_3c_1}{d_3} & & 1 \end{pmatrix} \xrightarrow{\iota_2\begin{pmatrix} 1 & \\ c_2/d_2 & 1 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ -\frac{b_3c_1}{d_3} & -\frac{2b_3c_2}{d_1d_2d_3} & 1 & \frac{b_3}{d_1d_3} \\ & -\frac{b_3c_1}{d_3} & & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ -\frac{b_3c_1}{d_3} & -\frac{2b_3c_2}{d_1d_2d_3} & 1 & \frac{b_3}{d_1d_3} \\ & -\frac{b_3c_1}{d_3} & & 1 \end{pmatrix} \xrightarrow{\iota_2\begin{pmatrix} d_2^{-1} & \\ & d_2 \end{pmatrix}} \begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ -\frac{b_3c_1d_2^2}{d_3} & -\frac{2b_3c_2d_2}{d_1d_3} & 1 & \frac{b_3}{d_1d_3} \\ & -\frac{b_3c_1d_2^2}{d_3} & & 1 \end{pmatrix}$$

The conjugations have added lower diagonal entries to this last matrix, but we can factor this final matrix as

$$\begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ & & 1 & \frac{b_3}{d_1d_3} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ -\frac{b_3c_1d_2^2}{d_3} & -\frac{2b_3c_2d_2}{d_1d_3} + \frac{b_3^2c_1d_2^2}{d_1d_3^2} & 1 & \\ & -\frac{b_3c_1d_2^2}{d_3} & & 1 \end{pmatrix} \tag{6.17}$$

to group all the upper triangular part of $\gamma(B_i)$ together on the left. These three matrices are then

$$\begin{pmatrix} 1 & & \frac{b_2}{d_2} & \\ & 1 & & \frac{b_2}{d_2} \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{b_1c_2}{d_1d_2} & & \frac{b_1}{d_1d_2^2} \\ & 1 & & \\ & & 1 & \frac{b_1c_2}{d_1d_2} \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{b_3}{d_1d_3} & & \\ & 1 & & \\ & & 1 & \frac{b_3}{d_1d_3} \\ & & & 1 \end{pmatrix},$$

and we see that this product takes of the form

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix},$$

the leftmost matrix in (6.16).

Now we pull all the lower triangular factors to the right in the same way, conjugating them by the diagonal matrices of the decomposition. We begin with the rightmost matrix, which is the lower triangular factor

$$\begin{pmatrix} 1 & & & \\ -\frac{c_3}{d_3} & 1 & & \\ & & 1 & \\ & & \frac{c_3}{d_3} & 1 \end{pmatrix}.$$

Next we have the middle factor's lower triangular matrix, which we need to conjugate past one diagonal matrix.

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{c_1}{d_1} & & & 1 \end{pmatrix} \xrightarrow{\iota_3\begin{pmatrix} d_3^{-1} & \\ & d_3 \end{pmatrix}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{c_1}{d_1 d_3^2} & & & 1 \end{pmatrix}$$

Then we have the left factor's lower triangular matrix, which gets conjugated by two diagonal matrices.

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ \frac{c_2}{d_2} & & 1 & \\ & \frac{c_2}{d_2} & & 1 \end{pmatrix} \xrightarrow{\iota_1\begin{pmatrix} d_1^{-1} & \\ & d_1 \end{pmatrix}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ \frac{c_2}{d_1 d_2} & & 1 & \\ & \frac{c_2}{d_1 d_2} & & 1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ \frac{c_2}{d_1 d_2} & & 1 & \\ & \frac{c_2}{d_1 d_2} & & 1 \end{pmatrix} \xrightarrow{\iota_3\begin{pmatrix} d_3^{-1} & \\ & d_3 \end{pmatrix}} \begin{pmatrix} 1 & & & \\ & 1 & & \\ \frac{c_2}{d_1 d_2} & & 1 & \\ & \frac{c_2}{d_1 d_2} & & 1 \end{pmatrix}$$

Finally, we have the lower triangular matrix factored from (6.17), which needs to be conjugated by all three diagonal matrices. The final result is below.

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ -\frac{b_3c_1d_2^2}{d_3} & -\frac{2b_3c_2d_2}{d_1d_3}+\frac{b_3^2c_1d_2^2}{d_1d_3^2} & 1 & \\ & -\frac{b_3c_1d_2^2}{d_3} & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ -\frac{b_3c_1}{d_1d_3} & -\frac{2b_3c_2d_3}{d_1d_2}+\frac{b_3^2c_1}{d_1} & 1 & \\ & -\frac{b_3c_1}{d_1d_3} & & 1 \end{pmatrix}$$

Now that the lower triangular matrices are grouped together, we take their product and show its form matches the right matrix in (6.16).

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ -\frac{b_3c_1}{d_1d_3} & -\frac{2b_3c_2d_3}{d_1d_2}+\frac{b_3^2c_1}{d_1} & 1 & \\ & -\frac{b_3c_1}{d_1d_3} & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ \frac{c_2}{d_1d_2} & & 1 & \\ & \frac{c_2}{d_1d_2} & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \frac{c_1}{d_1d_3^2} & & & 1 \end{pmatrix} \cdot$$
$$\cdot \begin{pmatrix} 1 & & & \\ -\frac{c_3}{d_3} & 1 & & \\ & & 1 & \\ & & \frac{c_3}{d_3} & 1 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ -\frac{c_3}{d_3} & 1 & & \\ * & -\frac{2b_3c_2d_3}{d_1d_2}+\frac{b_3^2c_1}{d_1} & 1 & \\ * & * & \frac{c_3}{d_3} & 1 \end{pmatrix}. \quad (6.18)$$

Lastly, in the middle of the factorization, the three diagonal matrices combine into

$$\begin{pmatrix} d_1^{-1}d_2^{-1}d_3^{-1} & & & \\ & \frac{d_3}{d_2} & & \\ & & \frac{d_2}{d_3} & \\ & & & d_1d_2d_3 \end{pmatrix} = \mathfrak{D}_{d_1,d_2,d_3}. \quad \blacksquare$$

So by Lemma 6.7, the argument of Θ_f in the Whittaker coefficient (6.12) can be written

$$\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \mathfrak{D}_{d_1,d_2,d_3} \begin{pmatrix} 1 & & & \\ -\frac{c_3}{d_3} & 1 & & \\ * & -\frac{2b_3c_2d_3}{d_1d_2}+\frac{b_3^2c_1}{d_1} & 1 & \\ * & * & \frac{c_3}{d_3} & 1 \end{pmatrix} w_0n_+(x_{ij}).$$

If we moved the lower triangular matrix across w_0, it becomes the upper triangular unipotent

$$\begin{pmatrix} 1 & \frac{c_3}{d_3} & * & * \\ & 1 & \frac{2b_3c_2d_3}{d_1d_2} - \frac{b_3^2c_1}{d_1} & * \\ & & 1 & -\frac{c_3}{d_3} \\ & & & 1 \end{pmatrix}.$$

Then, we can combine this with $n_+(x_{ij})$ and make linear substitutions for the x_{ij} to eliminate this matrix. The substitutions are additive and so don't change the measure of the integral, but the changes to x_{12} and x_{23} are recorded in the argument of the character ψ.

The resulting Whittaker coefficient is

$$\int_{\mathcal{R}} \int_{F_S^3} \sum_{d_i, c_i} \Theta_f \left(\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \mathfrak{D}_{d_1,d_2,d_3} w_0 n_+(x_{ij}) \right)$$

$$\cdot \psi \left(-m_1 \left(x_{12} - \frac{c_3}{d_3} \right) - m_2 \left(x_{23} - \frac{2b_3c_2d_3}{d_1d_2} + \frac{b_3^2c_1}{d_1} \right) \right) \prod_{j=2}^{4} dx_{1j} dx_{23}$$

$$= \sum_{d_i} \left[\sum_{c_i} \psi \left(m_1 \frac{c_3}{d_3} + m_2 \left(\frac{2b_3c_2d_3}{d_1d_2} - \frac{b_3^2c_1}{d_1} \right) \right) \right]$$

$$\cdot \int_{\mathcal{R}} \int_{F_S^3} \Theta_f \left(\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \mathfrak{D}_{d_i} w_0 n_+(x_{ij}) \right) \psi(-m_1 x_{12} - m_2 x_{23}) \prod_{j=2}^{4} dx_{1j} dx_{23}.$$

Now we remove the leftmost matrix in Θ_f's argument by the following logic.

Lemma 6.8. *For any $g \in \mathrm{Sp}_4(F_S)$,*

$$\Theta_f \left(\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} g \right) = \Theta_f(g).$$

Proof.

$$\begin{aligned}
\Theta_f\left(\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} g\right) &= \sum_{\gamma'} f\left(\iota'(\gamma')\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} g\right) \\
&= \sum_{\gamma'=\begin{pmatrix} a & b \\ c & d \end{pmatrix}} f\left(\begin{pmatrix} 1 & * & * & * \\ & a & b & * \\ & c & d & * \\ & & & 1 \end{pmatrix} g\right) \\
&= \sum_{\gamma'} f\left(\begin{pmatrix} 1 & * & * & * \\ & 1 & & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \iota'(\gamma') g\right) = \Theta_f(g). \quad \blacksquare
\end{aligned}$$

Combining all the work done so far, the (m_1, m_2)-Whittaker coefficient of our Eisenstein series E_f can be written

$$\sum_{d_i} H_\psi(d_1, d_2, d_3) \int_{\mathcal{R}} \left[\int_{F_S^3} \Theta_f\left(\mathfrak{D}_{d_i} w_0 n_+(x_{ij})\right) \psi(-m_1 x_{12} - m_2 x_{23}) \prod_{j=2}^{4} dx_{1j} \right] dx_{23}, \tag{6.19}$$

where

$$H_\psi(d_1, d_2, d_3) = \sum_{c_i} \psi\left(m_1 \frac{c_3}{d_3} + m_2\left(\frac{2b_3 c_2 d_3}{d_1 d_2} - \frac{b_3^2 c_1}{d_1}\right)\right).$$

We notate the quantity (6.19) by $a_f^{\mathrm{Sp}(4)}(m_1, m_2; \psi)$.

6.3 The Recursion

Now we reduce the Whittaker calculation over $\mathrm{Sp}_4(F_S)$ to one over $\mathrm{SL}_2(F_S)$. For $f \in \pi(s_1, s_2)$, $m_1 \in \mathfrak{O}_S$, and $g' \in \mathrm{SL}_2(F_S)$, we define $f^{(m_1)}_{d_1,d_2,d_3}(g')$ by the following integral:

$$f_{d_i}^{(m_1)} = \int\limits_{F^3} f\left(\iota'(g')\mathfrak{D}_{d_i}\begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ -1 & & & \end{pmatrix} n_+(x_{12}, x_{13}, x_{14}, 0)\right)\bar{\psi}(m_1 x_{12})dx_{1j}.$$

By (6.1), we have that

$$f_{d_1,d_2,d_3}^{(m_1)}\left(\begin{pmatrix} t_1 & * \\ & t_1^{-1} \end{pmatrix} g'\right) = |t_1|^{2s_1} f(g'). \tag{6.20}$$

Following the definition of $\pi(s_1, s_2)$ to define the rank one $\mathrm{SL}_2(F_S)$ representation $\pi(s_1)$, we see that (6.20) implies that $f_{d_1,d_2,d_3}^{(m_1)} \in \pi(s_1)$. Then we can define our Eisenstein series over $\mathrm{SL}_2(F_S)$ in the natural way. For $g' \in \mathrm{SL}_2(F)$ and $f \in \pi(s_1)$,

$$E_f^{\mathrm{SL}(2)}(g') = \sum_{\gamma' \in B_{\mathrm{SL}(2)}(\mathfrak{O}_S)\backslash \mathrm{SL}_2(\mathfrak{O}_S)} f(\gamma' g').$$

We introduce the two notations

$$w_0' = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad \text{and} \qquad n_+'(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}.$$

Then, given a choice of ψ, we may define the Whittaker coefficient of $E_f^{\mathrm{SL}(2)}$ indexed by $m_1 \in \mathfrak{O}_S$ as

$$a_f^{\mathrm{SL}(2)}(m_1;\psi) := \int\limits_{\mathcal{R}} E_f^{\mathrm{SL}(2)}(w_0' n_+'(x))\psi(-m_1 x)dx. \tag{6.21}$$

Theorem 6.9. *The rank two Whittaker coefficients* (6.19) *can be written in terms of their rank one counterparts in the following way.*

$$a_f^{\mathrm{Sp}(4)}(m_1, m_2;\psi) = \sum_{d_1,d_2,d_3 \in \mathfrak{O}_S\backslash\{0\}/\mathfrak{O}_S^\times} H_\psi(d_i) a_{f_{d_1,d_3,d_2}^{(m_1)}}^{\mathrm{SL}(2)}\left(\frac{d_3^2}{d_2^2} m_2;\psi\right). \tag{6.22}$$

Before we prove this, we observe the following calculation.

Lemma 6.10.

$$E^{\mathrm{SL}_2}_{f^{(m_1)}_{d_1,d_2,d_3}}\left(w_0'n_+'(x_{23})\right)$$
$$= \int\limits_{F^3} \Theta_f\left(\mathfrak{D}_{d_1,d_3,d_2} w_0 n_+(x_{12},x_{13},x_{14},\frac{d_2^2}{d_3^2}x_{23})\right)\psi(-m_1x_{12})\prod_{j=2}^{4} dx_{1j}$$

Proof. Substituting in our earlier definitions, we have

$$E^{\mathrm{SL}_2}_{f^{(m_1)}_{d_1,d_2,d_3}}\left(w_0'n_+'(x_{23})\right) = \sum_{\gamma'\in B_{\mathrm{SL}(2)}(\mathfrak{O}_S)\backslash \mathrm{SL}_2(\mathfrak{O}_S)} f^{(m_1)}_{d_1,d_2,d_3}\left(\gamma' w_0'n_+'(x_{23})\right)$$
$$= \sum_{\gamma'}\int\limits_{F^3} f\left(\iota'\left(\gamma' w_0' n_+(x_{23})\right)\mathfrak{D}_{d_i}\iota_1(w_0)n_+(x_{12},x_{13},x_{14},0)\right)\bar{\psi}(m_1x_{12})\prod_{j=2}^{4}dx_{1j}$$
$$= \int\limits_{F^3}\Theta_f\left(\mathfrak{D}_{d_1,d_3,d_2} w_0 n_+(x_{12},x_{13},x_{14},\frac{d_2^2}{d_3^2}x_{23})\right)\psi(-m_1x_{12})dx_{1j}. \quad \blacksquare$$

Remark 6.11. The subscripts d_2 and d_3 get switched in $\mathfrak{D}_{d_i}$ because the middle two entries are conjugated by w_0'. Also, the coefficient of x_{23} comes from conjugation by $\mathfrak{D}$.

Proof of Theorem. Substituting Lemma 6.10 into (6.19) gives the following form of the rank two coefficient.

$$a_f^{\mathrm{Sp}(4)}(m_1,m_2;\psi) = \sum_{d_i} H_\psi(d_i)\cdot\int\limits_{\mathcal{R}} E^{\mathrm{SL}(2)}_{f^{(m_1)}_{d_1,d_3,d_2}}\left(w_0'n_+'\left(\frac{d_2^2}{d_3^2}x_{23}\right)\right)\psi(-m_2x_{23})dx_{23}.$$

Now we use the substitution $x_{23}\longrightarrow \frac{d_3^2}{d_2^2}x_{23}$. This variable change only alters the region over which we are averaging from $F_S/\mathfrak{O}_S$ to $F_S/((\frac{d_3}{d_2})^2\mathfrak{O}_S)$. So these two integrations are equal, as long as they are both defined.

$$\int_{(d_3^2d_2^{-2}\mathfrak{O})\backslash F}(-*-)d\left(\frac{d_3^2}{d_2^2}x_{23}\right) = \int_{\mathfrak{O}\backslash F}(-*-)dx_{23}$$

With careful inspection of H, we may see that $d_2^2|m_2d_3^2$ or this character sum is trivial, so we know that the second integral is well-defined. So we can finally write the Whittaker coefficient as

$$a_f^{\mathrm{Sp}(4)}(m_1, m_2; \psi) = \sum_{d_i} H_\psi(d_i) a^{\mathrm{SL}(2)}_{f^{(m_1)}_{d_1,d_3,d_2}} \left(\frac{d_3^2}{d_2^2} m_2; \psi \right). \tag{6.23}$$

■

The Casselman-Shalika formula gives Fourier-Whittaker coefficients of Eisenstein series as characters. Hence, applying the Casselman-Shalika formula we have

Theorem 6.12.

$$\chi^{\mathrm{SO}(2r+1)}(m_1, m_2) = \sum_{d_i} H_\psi(d_i) \chi^{\mathrm{GL}(2)} \left(\frac{d_3^2}{d_2^2} m_2 \right), \tag{6.24}$$

where χ^G are highest weight characters for the complex Lie groups $G(\mathbb{C})$.

6.4 Higher Rank Cases

Above we have reduced the Whittaker calculation from Sp_4 to $\mathrm{Sp}_2 = \mathrm{SL}_2$. The natural follow-up question is whether higher even-rank symplectic groups can be dealt with in the same manner. In short, the answer is yes, although the calculation becomes more complicated at certain steps.

We begin with $\mathrm{Sp}_{2r}(F_S)$ with the following natural embedding ι' of $\mathrm{Sp}_{2(r-1)}(F_S)$:

$$\iota'(g') = \begin{pmatrix} 1 & & \\ & \begin{pmatrix} g' \end{pmatrix} & \\ & & 1 \end{pmatrix},$$

for $g' \in \mathrm{Sp}_{2(r-1)}(F_S)$. We have the natural generalization to $\pi(s_1, s_2, \cdots, s_r)$, the right regular representation of $\mathrm{Sp}_{2r}(F_S)$ on the set of smooth functions f with

$$f(bg) = \left(\prod_{i=1}^{r} |t_i|^{\sum_{j=1}^{i} 2s_j} \right) f(g),$$

for $b \in B_{\mathrm{Sp}(2r)}(\mathfrak{O}_{\mathfrak{S}})$ with diagonal elements $(t_r, t_{r-1}, \ldots, t_1, t_1^{-1}, t_2^{-1}, \ldots, t_r^{-1})$.

This leads to the definition of $E_f^{\mathrm{Sp}(2r)} : \mathrm{Sp}_{2r}(F_S) \longrightarrow \mathbb{C}$ by

$$E_f^{\mathrm{Sp}(2r)}(g) = \sum_{\gamma \in B_{\mathrm{Sp}(2r)}(\mathfrak{O}_S) \backslash \mathrm{Sp}_{2r}(\mathfrak{O}_S)} f(\gamma g).$$

Just as in the rank two case, this Eisenstein series has Whittaker coefficient defined as the integral

$$a_f^{\mathrm{Sp}(2r)}(m_1,\dots,m_r;\psi) := \int_{\mathcal{R}^{r^2}} E_f^{\mathrm{Sp}(2r)}\left(w_0 n_+(x_{ij})\right)\bar{\psi}\left(\sum_{k=1}^{r} m_k x_{k,k+1}\right)\prod_{\substack{1\le i<j\le 2r\\ i+j\le 2r+1}} dx_{ij}.$$

In this formula we define w_0 as the long element of Sp_{2r} with nonzero entries only along the opposite diagonal which are -1 in the bottom left quadrant and 1 in the top right quadrant. Also, $n_+(x_{ij})$, for i,j running over the indices $1 \le i,j \le 2r$ with $i+j \le 2r+1$, is the unique upper triangular unipotent matrix in $\mathrm{Sp}_{2r}(F_S)$ with the x_{ij} in its ith row and jth column.

As in the work above, we use the function

$$\Theta_f^{\mathrm{Sp}(2r)}(g) = \sum_{\gamma\in P_{\mathrm{Sp}(2r)}(\mathfrak{O}_S)\backslash \mathrm{Sp}_{2r}(\mathfrak{O}_S)} f(\gamma g),$$

where $P_{\mathrm{Sp}(2r)}(R)$ is the maximal parabolic subgroup of $\mathrm{Sp}_{2r}(R)$ defined as

$$P_{\mathrm{Sp}(2r)}(R) = \left\{(a_{ij}) \in \mathrm{Sp}_{2r}(R) \mid a_{i,1} = 0 \text{ for } i > 1, \text{ and } a_{2r,j} = 0 \text{ for } j < 2r\right\}.$$

We may then substitute $\Theta_f^{\mathrm{Sp}(2r)}$ into $E_f^{\mathrm{Sp}(2r)}$ by summing over the lowest row $(B_{2r}, B_{2r-1},\dots,B_1)$ of γ, using a higher rank version of Proposition 6.1. Lemma 6.2 and Proposition 6.5 also easily generalize, so we know that we may drop the terms of the sum with $B_1 = 0$, and expand the integrals over x_{1j} to all of F_S which restricting the sums over B_i to be modulo B_1.

Now we want to factor $\gamma(B_i)$ into SL_2 embeddings to generalize Lemma 6.3. As in Remark 6.4, we use a decomposition of w_0 into simple reflections to guide the order of the factorization. Our simple reflections are

$$\sigma_1 = \begin{pmatrix} I_{r-1} & & & \\ & & -1 & \\ & 1 & & \\ & & & I_{r-1}\end{pmatrix}, \qquad \sigma_i = \begin{pmatrix} I_{r-i} & & & & & & \\ & & 1 & & & & \\ & -1 & & & & & \\ & & & I_{2i-4} & & & \\ & & & & & -1 & \\ & & & & 1 & & \\ & & & & & & I_{r-i}\end{pmatrix}$$

for $i = 2, 3, \ldots, r$. A convenient decomposition of w_0 is

$$w_0 = \prod_{i=1}^{r} \left(\sigma_{r+1-i}\sigma_{r-i} \cdots \sigma_2\sigma_1\sigma_2 \cdots \sigma_{r-i}\sigma_{r+1-i}\right).$$

So, for example, if $r = 1, 2, 3$ the decompositions are

$$w_0 = \sigma_1 \qquad w_0 = (\sigma_2\sigma_1\sigma_2)\sigma_1 \qquad w_0 = (\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3)(\sigma_2\sigma_1\sigma_2)\sigma_1.$$

By checking which roots are negated by partial products of the above decomposition, we are left with the following ordering of the relevant roots (those including α_r):

$$\sum_{i=1}^{r} \alpha_i, \sum_{i=1}^{r} \alpha_i + \alpha_2, \sum_{i=1}^{r} \alpha_i + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \sum_{i=2}^{r} 2\alpha_i,$$
$$\alpha_r, \alpha_{r-1} + \alpha_r, \ldots, \sum_{i=2}^{r} \alpha_i. \quad (6.25)$$

To convert the ordering (6.25) to a factorization of γ, we look at the relevant SL_2 embeddings. We index them by $i = 1, 2, \ldots, 2r-1$. For $i = 1$ we have

$$\iota_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & b \\ & I_{2r-2} & \\ c & & d \end{pmatrix}.$$

For $i = 2, 3, \ldots r$ we have

$$\iota_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & & & b & & \\ & I_{i-2} & & & & & \\ & & a & & & & b \\ & & & I_{2r-2i} & & & \\ c & & & & d & & \\ & & & & & I_{i-2} & \\ & & c & & & & d \end{pmatrix}.$$

Finally, for $i = r+1, r+2, \dots, 2r-1$, we have

$$\iota_i \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & & -b & & & & \\ & I_{2r-1-i} & & & & & \\ -c & & d & & & & \\ & & & I_{2i-2r-2} & & & \\ & & & & a & & b \\ & & & & & I_{2r-1-i} & \\ & & & & c & & d \end{pmatrix}.$$

The ordering of roots (6.25) suggests the following ordering of factors:

$$\gamma(B_i) = \iota_r(-)\iota_{r-1}(-)\cdots\iota_1(-)\iota_{2r-1}(-)\iota_{2r-2}(-)\cdots\iota_{r+1}(-). \tag{6.26}$$

Remark 6.13. The arguments of each ι_i in (6.26) is the 4-tuple $(a_i, b_i, c_i, d_i) \in F_S^4$ with $a_i d_i - b_i c_i = 1$.

The values for these matrix entries are determined by the B_i up to choices similar to the ones found in rank two (see Lemma 6.3), though the relationships get more complicated as r gets large. We have

$$B_1 = \prod_{j=1}^{2r-1} d_j,$$

and for $1 < i < r$,

$$B_i = c_{2r+1-i} \prod_{j=1}^{r} d_j \prod_{k=1}^{i-2} d_{2r-k}.$$

These fix the values of d_i and c_i for $i > r$, and also it fixes the product $d_1 d_2 \cdots d_r$.

The remaining c_i and d_i ($i \leq r$) can be determined by the B_i and choices of a_i and b_i for ($i > r$). First, we have

$$\begin{aligned} d_{2r-1}(d_{2r-2}(\cdots(d_{r+1}B_{2r} + c_{r+1}B_{r+1}) + \cdots) + c_{2r-2}B_{2r-2}) + c_{2r-1}B_{2r-1} \\ = c_1 d_2 d_3 \cdots d_r, \end{aligned}$$

which fixes c_1, d_1, and $d_2 d_3 d_4$. Then we take the left hand side of the above equation and replace d_{2r-1} with b_{2r-1} and c_{2r-1} with a_{2r-1}, and it becomes equal to $c_2 d_3 d_4 \cdots d_r$, which fixes c_2, d_2, and $d_3 d_4 \cdots d_r$. Then the layers are peeled off, with b_i and a_i replacing d_i and c_i respectively, and we get $c_{2r+1-i} d_{2r+2-i} d_{2r+3-i} \cdots d_{2r-1}$ which fixes

c_{2r+1-i}, d_{2r+1-i}, and the product $d_{2r+2-i}d_{2r+3-i}\cdots d_r$, for $i = 2r-2, 2r-3, \ldots r+1$. The final equation is thus

$$b_{r+1}B_{2r} + a_{r+1}B_{r+1} = c_r.$$

Once this factorization is determined, the same steps of determining the moduli over which the c_i are summed and breaking the factorization into lower triangular, diagonal, and upper triangular parts determine the exponential sum in front of the lower rank Whittaker integral. We content ourselves with onlly the explicit rank two computation here.

6.5 The metaplectic calculation

Another way in which the above calculation can be generalized is by considering metaplectic covers of Sp_4, or more generally Sp_{2r}. For some fixed $n > 1$, let μ_{2n} be the $2n$th roots of unity in F, and assume $\#(\mu_{2n}) = 2n$. Also, to S our set of bad places S, include every place v dividing n. Then an n-fold metaplectic cover is constructed by defining a cocycle, $\sigma : \mathrm{Sp}_{2r}(F_S)^2 \longrightarrow \mu_n$, and then defining the product on $\mathrm{Sp}_{2r}(F_S) \times \mu_n$ by

$$(g, \zeta) \cdot (g', \zeta') = (gg', \zeta\zeta'\sigma(g, g')).$$

The group $\mathrm{Sp}_{2r}(F_S) \times \mu_n$ endowed with this product is the metaplectic group $\widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S)$. This construction is explained in detail in [**?**] and [12], but because this is only a discussion of a direction of generalization we do not include the explicit cocycle here.

The definition of the metaplectic Eisenstein series utilizes the fact that the group $\mathrm{Sp}_{2r}(\mathfrak{O}_S)$ lifts to $\widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S)$. This lift is a homomorphism

$$\begin{aligned} \mathrm{Sp}_{2r}(\mathfrak{O}_S) &\longrightarrow \widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S) \\ g &\longrightarrow (g, \kappa(g)), \end{aligned}$$

for $\kappa : \mathrm{Sp}_{2r}(\mathfrak{O}_S) \longrightarrow \mu_n$ the Kubota symbol. The image of this map is denoted $\mathrm{Sp}_{2r}^*(\mathfrak{O}_S)$, a subgroup of $\widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S)$. Similarly, any subgroup of $\mathrm{Sp}_{2r}(\mathfrak{O}_S)$ can be lifted by the same map, and its image will be denoted with a superscript $*$.

We also define the section

$$\begin{aligned} \boldsymbol{s} : \mathrm{Sp}_{2r}(F_S) &\longrightarrow \widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S) \\ g &\longrightarrow (g, 1). \end{aligned}$$

Note that this is not a homomorphism for any $n > 1$, but we have $\boldsymbol{s}(g)\boldsymbol{s}(h) = \sigma(g, h)\boldsymbol{s}(gh)$.

We say a function $f : \widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S) \longrightarrow \mathbb{C}$ is *genuine* if

$$f(\zeta\tilde{g}) = \zeta f(\tilde{g})$$

for all $\tilde{g} \in \widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S)$ and $\zeta \in \mu_n$.

Now we define our metaplectic Eisenstein series. For $\tilde{g} \in \widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S)$, let

$$\widetilde{E}_f^{\mathrm{Sp}(2r),n}(\tilde{g}) = \sum_{\gamma \in B_{\mathrm{Sp}(2r)}(\mathfrak{O}_S) \backslash \mathrm{Sp}_{2r}(\mathfrak{O}_S)} \kappa(\gamma) f(\boldsymbol{s}(\gamma)\tilde{g}).$$

Here f is an element in $\pi^{(n)}(s_1, \ldots, s_r)$, the right regular representation of $\widetilde{\mathrm{Sp}}_{2r}^{(n)}(F_S)$ of smooth genuine functions satisfying the following left translation condition.

For the nth-order Hilbert symbol $(-,-)_S = \prod_{v \in S}(-,-)_v$, an *isotropic* subgroup of $F_S^\times$ is a subgroup with $(x, y)_S = 1$ for all pairs of elements x, y in the subgroup. Then, we consider the maximal isotropic subgroup of $F_S^\times$, $\Omega = \mathfrak{O}_S^\times (F_S^\times)^n$. Then the left translation condition for $f \in \pi^{(n)}(s_1, \ldots, s_r)$ is that

$$f\left(\boldsymbol{s}\begin{pmatrix} t_r & * & \cdots & * & * & \cdots & * \\ & t_{r-1} & \cdots & * & * & \cdots & * \\ & & \ddots & \vdots & \vdots & \ddots & \vdots \\ & & & t_1 & * & \cdots & * \\ & & & & t_1^{-1} & \cdots & * \\ & & & & & \ddots & \vdots \\ & & & & & & t_r^{-1} \end{pmatrix}\tilde{g}\right) = \prod_{i=1}^{r} |t_i|^{\sum_{j=1}^{i} 2s_j} f(\tilde{g}),$$

for $t_i \in \Omega$.

In addition we have the trivial lift of unipotent upper triangular matrices of $\mathrm{Sp}_{2r}(F_S)$, because the cocycle can be defined to be trivial on these matrices. This lift allows us to define our Whittaker coefficient as the integral

$$\int_{\mathcal{R}^{r^2}} \widetilde{E}_{\tilde{f}}^{\mathrm{Sp}(2r),n}((w_0,1)(n_+(x_{ij}),1))\bar{\psi}\left(\sum_{i=1}^{r} m_i x_{i,i+1}\right)\prod dx_{ij}.$$

From here, the calculation follows exactly as it did in the nonmetaplectic case, except that at each step in which matrices are being multiplied or moved past one another, the cocycle must be evaluated. This leads to the accumulation of Hilbert symbols, and the hope (and expectation) is that a careful account of this will result in a more complicated version of our earlier exponential sum H_ψ but now twisted by nth power residue symbols. This has been carried out for metaplectic covers of $\mathrm{SL}_{r+1}(F_S)$ by Bump, Brubaker, and Friedberg [3]. If the higher rank complexities are surmountable, a generalization of this work could determine a general recursion leading to new explicit formulas for $\widetilde{\mathrm{Sp}}_{2r}^{(n)}$-Eisenstein series Whittaker coefficients.

Chapter 7

Metaplectic Hecke operators on $\mathrm{GL}_3(F)$

In this final chapter we carry out a second computation of Whittaker coefficients, this time of an arbitrary metaplectic form ϕ on the six-cover of GL_3. Our approach is to explicitly compute the Hecke operator action on ϕ, and then use that the form is an eigenvector of the action to observe relations on its Whittaker coefficients. This method comes from the work of Hoffstein [9], who computed the action for the lift of the matrix

$$\gamma_1 = \begin{pmatrix} p^6 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

We extend this calculation to the other generator of the Hecke algebra, namely the lift of

$$\gamma_2 = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & 1 \end{pmatrix}.$$

We show this second action gives only redundant information about the structure of the coefficients of ϕ, by demonstrating that the γ_2 action may be reduced to the γ_1 action. In fact, we see that the difference between the two actions amounts to a choice of ordering of the simple positive roots of GL_3.

7.1 Setup

We begin with a number field F and its ring of integers $\mathfrak{O}$. We also assume the constructions of the metaplectic group $\widetilde{G}$, the n cover of $G = \mathrm{GL}_r(F)$ for integers $r, n > 1$. By the work of Matsumoto [?] and Kubota [?], this construction admits a lift of $\Gamma = \mathrm{SL}_r(\mathfrak{O})$, which we will write Γ^*. This lift can be explicitly determined by the Kubota symbol. When $r = 2$, the Kubota symbol has the conveniently simple expression

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \left(\frac{c}{d}\right)_n & \text{if } c \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Here $(\frac{c}{d})_n$ is the nth-power residue symbol, which we assume to have perfect reciprocity in F.

We will be working with the 6-cover of $\mathrm{GL}_3(F)$, so we would like to know how the Kubota symbol generalizes to the $r = 3$ case. Fortunately, due to the result of Banks-Levy-Sepanski [1] concerning block compatibility, the three embeddings of $\mathrm{SL}_2(\mathfrak{O})$ into $\mathrm{SL}_3(\mathfrak{O})$ of matrices of the forms

$$\begin{pmatrix} * & * & \\ * & * & \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} * & & * \\ & 1 & \\ * & & * \end{pmatrix} \qquad \begin{pmatrix} 1 & & \\ & * & * \\ & * & * \end{pmatrix} \tag{7.1}$$

preserve the Kubota symbols of the matrices. Since the above matrices generate Γ, knowledge of their Kubota symbols along with knowledge about the cocycle σ of the metaplectic group gives the lift of every element of Γ.

7.2 Hecke Operators

We now describe the Hecke algebra action on the space of metaplectic forms. Observe that we can act on functions on $\widetilde{G}$ by left translation; that is, for f a function and $g, h \in \widetilde{G}$, the action of g on f is given by

$$(f|_g)(h) = f(gh).$$

Also, we call a function on $\widetilde{G}$ *genuine* if

$$f(\zeta g) = \zeta f(g)$$

for all $g \in \widetilde{G}$ and $\zeta \in \mu_n$. Metaplectic forms are then defined to be smooth, genuine functions on $\widetilde{G}$ which are automorphic with respect to the action of Γ^* described above.

For an element g of $\widetilde{G}$, we write the double coset of $\Gamma^* g \Gamma^*$ as a disjoint union of right cosets

$$\Gamma^* g \Gamma^* = \bigcup_{i=1}^{M} \Gamma^* g_i,$$

for some choice of coset representatives $g_i \in \tilde{G}$. Then the action of the Hecke operator associated to g, T_g, on a metaplectic form ϕ is given by

$$T_g \phi = \sum_{i=1}^{M} \phi|_{g_i}.$$

We note that this is well-defined because automorphicity implies that it is independent of the choices of representatives. Also, $T_g\phi$ is a metaplectic form itself because

$$(T_g\phi)(\gamma h) = \sum_{i=1}^{M} \phi(g_i \gamma h).$$

Now $g_i\gamma \in \Gamma^* g \Gamma^*$ for each i, and this multiplication just permutes the representatives because

$$\Gamma^* g_i \gamma = \Gamma^* g_j \gamma \Longrightarrow \Gamma^* g_i = \Gamma^* g_j \Longrightarrow i = j.$$

Since the set $\{g_i\gamma\}$ is another choice of right coset representatives, we have shown that

$$(T_g\phi)(\gamma h) = (T_g\phi)(h),$$

and thus T_g preserves automorphicity.

7.3 Computing T_{γ_1}

For any element of $\widetilde{G}$ of the form $g = (h, 1)$, we write $T_{(h,1)} = T_h$ to denote the associated Hecke operator. The first computation we discuss is determining the effect of T_{γ_1} on the space of metaplectic functions, for

$$\gamma_1 = \begin{pmatrix} p^6 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

with p a fixed prime in $\mathfrak{O}$. The steps for this computation are as follows.

1. First we find a convenient set of right coset representatives of $\Gamma^*\gamma_1\Gamma^*$ with which to compute.

2. Next we find the Kubota symbol for these coset representatives.

3. Finally we calculate the Whittaker coefficients of $T_{\gamma_1}\phi$ in terms of those of ϕ.

Each of these are detailed below.

7.3.1 Right coset representatives

Throughout this section we ignore the metaplecticity of Γ^*, and only concern ourselves with determining the right cosets of Γ in $\Gamma\gamma_1\Gamma$. The root of unity attached to each coset representative will be determined in the following section.

Proposition 7.1. *The set*

$$H := \left\{ \eta = \begin{pmatrix} p^i & a & c \\ & p^j & b \\ & & p^k \end{pmatrix} \middle| \begin{array}{l} i+j+k=6,\ i,j,k \geq 0,\ 0 \leq a < p^j, \\ 0 \leq b, c < p^k,\ \text{and the } p\text{-rank of } \eta \text{ is two} \end{array} \right\} \tag{7.2}$$

is complete set of right coset representatives for $\Gamma\gamma_1\Gamma$.

Note that the inequalities on a, b, c require them to run over a fixed set of residues of the moduli p^j and p^k in $\mathfrak{O}$, in the case that this set of integers is not $\mathbb{Z}$.

Proof. To show H is a sufficient set of representatives we must have that any element of $\Gamma\gamma_1\Gamma$ can be multiplied on the left by an appropriate element of Γ to get it in the form of an element of H. If $g \in \Gamma\gamma_1\Gamma$, then we have $\det g = p^6$, $g \in \mathrm{GL}_3(F)$ with entries in $\mathfrak{O}$, and g has p-rank 2. If $g = (a_{i,j})$ for $1 \leq i, j \leq 3$, we can act on the left by a matrix $h = (b_{i,j})$ constructed as follows.

First, let the vector that is the bottom row of h, $(b_{31}, b_{32}, b_{33}) = b_{3j}$, be in the direction of the cross product of the first two columns of g:

$$b_{31} = \frac{a_{11}a_{22} - a_{12}a_{21}}{d},\ b_{32} = \frac{a_{12}a_{31} - a_{11}a_{32}}{d},$$

$$b_{33} = \frac{a_{21}a_{32} - a_{22}a_{31}}{d},$$

where d just divides out the common divisor of the actual cross product. Then the vector b_{3j} is perpendicular to both vectors a_{i1} and a_{i2} (explanation of notation: to find the vector, let the unknown subscript run over the set $\{1,2,3\}$ to obtain its coordinates). Now, consider the sublattice of $\mathfrak{O}^3$ of vectors perpendicular to the vector a_{i1}. One element of this lattice is b_{3j}. Another element would be $a_{i1} \times b_{3j}$, which is perpendicular to the first. Thus this is a lattice of rank exactly two. Because we divided b_{3j} out by its gcd, the vector can serve as the side of a fundamental parallelogram of this 2-lattice. Choose the other side of the fundamental parallelogram, and let b_{2j} be the coordinates of that vector.

So we have chosen the bottom row of h to be perpendicular to the first two columns of g and the middle row of h to be perpendicular to the first column of g. This implies that hg is upper triangular. Also, because the bottom two rows of h generate the lattice of $\mathfrak{O}^3$ points that are in the plane they generate in F^3, we can choose a third vector that forms a fundamental parallelopiped for the lattice $\mathfrak{O}^3$ and fix this (or its negative) as the first row of h. This ensures that h has determinant 1, so $h \in \Gamma$. So we have picked an $h \in \Gamma$ such that hg is upper triangular.

Now we note that because $\det hg = p^6$, we must have that the diagonal elements are powers of p exactly as described in (7.2), the definition of H. Finally, by multiplying on the left by matrices of the forms (and in the order)

$$\begin{pmatrix} 1 & r & \\ & 1 & \\ & & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & & \\ & 1 & s \\ & & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{pmatrix} \tag{7.3}$$

we can ensure that the diagonal elements remain fixed while terms above the diagonal are shifted to a particular set of coset representatives modulo the diagonal elements directly beneath them. We first shift a because using the r-shift of the first matrix can alter the value of c modulo p^k, and this is why we must be careful to pick a fixed set of residues modulo the diagonal elements for any given values of i, j, and k.

Now, to show that every element of H is necessary, suppose $\eta_1, \eta_2 \in H$ have that $\gamma\eta_1 = \eta_2$ for $\gamma \in \Gamma$. We can then write

$$\eta_m = \begin{pmatrix} p^{i_m} & a_m & c_m \\ & p^{j_m} & b_m \\ & & p^{k_m} \end{pmatrix} \text{ for } m = 1,2 \text{ and then}$$

$$\eta_m^{-1} = \begin{pmatrix} p^{-i_m} & -p^{-i_m-j_m}a & -p^{-i_m-k_m}c + p^{-6}ab \\ & p^{-j_m} & -p^{-j_m-k_m}b \\ & & p^{-k_m} \end{pmatrix}.$$

Then $\gamma = \eta_2\eta_1^{-1}$ is upper triangular, so is of the form

$$\begin{pmatrix} p^{i_2-i_1} & -p^{i_2-i_1-j_1}a_1 + p^{-j_1}a_2 & -p^{i_2-i_1-k_1}c_1 + p^{i_2-6}a_1b_1 - p^{-j_1-k_1}b_1a_2 + p^{-k_1}c_2 \\ & p^{j_2-j_1} & -p^{j_2-j_1-k_1}b_1 + p^{-k_1}b_2 \\ & & p^{k_2-k_1} \end{pmatrix}.$$

Since this is in Γ, we immediately get that $i_1 = i_2, j_1 = j_2$, and $k_1 = k_2$. Removing the subscripts on these terms and canceling, we get

$$\gamma = \begin{pmatrix} 1 & p^{-j}(a_2 - a_1) & p^{-k}(c_2 - c_1) + p^{-j-k}b_1(a_1 - a_2) \\ & 1 & p^{-k}(b_2 - b_1) \\ & & 1 \end{pmatrix} \in \Gamma.$$

So we have that $a_1 \equiv a_2 \pmod{p^j}$, which implies that $a_1 = a_2$, since we fixed our residues. Now we see that the second summand in the top right entry disappears, so we also simply get that $c_1 \equiv c_2 \pmod{p^k}$ and $b_1 \equiv b_2 \pmod{p^k}$. Thus $c_1 = c_2$ and $b_1 = b_2$, and so $\eta_1 = \eta_2$. ■

7.3.2 Computing the Kubota symbol

Now we must consider the metaplectic group in our calculation. We know how to represent the right cosets of $\Gamma\gamma_1\Gamma$, but we actually need to know how to represent the right cosets of $\Gamma^*(\gamma_1, 1)\Gamma^*$. This means each right coset representative above needs to have a particular sixth root of unity attached to it. For any $\eta \in H$, let ζ_η denote the corresponding root of unity. To find the particular root of unity associated to any given matrix, we can build the matrix from γ_1 by multiplying on the left and right by block matrices whose Kubota symbols are easy to determine. We start with the most general form of η, an element of H:

$$\eta = \begin{pmatrix} p^i & a & c \\ & p^j & b \\ & & p^k \end{pmatrix},$$

and we assume that $\min\{i, j, k\} > 0$, which implies that $p \nmid a$ and $p \nmid b$ from the p-rank restriction on η. In the below computations, x, y, x', y' are always chosen to make the matrix have determinant 1. We begin with

$$\begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & xp^k \end{pmatrix} \gamma_1 \begin{pmatrix} x & & y \\ & 1 & \\ -p^{i+j} & & -b \end{pmatrix} = \begin{pmatrix} p^{i+j} & & b \\ & 1 & \\ & & p^k \end{pmatrix}.$$

Now we can use

$$\begin{pmatrix} & -1 & \\ 1 & x'p^j & \\ & & 1 \end{pmatrix} \begin{pmatrix} p^{i+j} & & b \\ & 1 & \\ & & p^k \end{pmatrix} \begin{pmatrix} x' & y' & \\ -p^i & cb'p^j - a & \\ & & 1 \end{pmatrix} = \begin{pmatrix} p^i & a - cb'p^j & \\ & p^j & b \\ & & p^k \end{pmatrix}$$

for b' an inverse of b modulo p^k. So finally we may left multiply twice:

$$\begin{pmatrix} 1 & & c(1-bb')p^{-k} \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & cb' & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} p^i & a - cb'p^j & \\ & p^j & b \\ & & p^k \end{pmatrix} = \begin{pmatrix} p^i & a & c \\ & p^j & b \\ & & p^k \end{pmatrix}.$$

The only block matrices with nontrivial Kubota symbols are the first two matrices that were multiplied on the right, their Kubota symbols are

$$\begin{aligned} \kappa\left(\begin{pmatrix} x & & y \\ & 1 & \\ -p^{i+j} & & -b \end{pmatrix}\right) \kappa\left(\begin{pmatrix} x' & y' & \\ -p^i & cb'p^j - a & \\ & & 1 \end{pmatrix}\right) &= \left(\frac{-p^{i+j}}{-b}\right)\left(\frac{-p^i}{cb'p^j - a}\right) \\ &= \left(\frac{b}{p}\right)^{i+j}\left(\frac{a}{p}\right)^{i}. \end{aligned}$$

Now we fix $i = 0$, and assume $\min\{j, k\} > 0$, so $p \nmid b$. Then we have

$$\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \gamma_1 \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & p^6 & \\ & & 1 \end{pmatrix},$$

and then we get that

$$\begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & xp^k \end{pmatrix} \begin{pmatrix} 1 & & \\ & p^6 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & x & y \\ & -p^j & -b \end{pmatrix} = \begin{pmatrix} 1 & & \\ & p^j & b \\ & & p^k \end{pmatrix}.$$

So we can right multiply to get the matrix:

$$\begin{pmatrix} 1 & & \\ & p^j & b \\ & & p^k \end{pmatrix} \begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & c \\ & p^j & b \\ & & p^k \end{pmatrix}.$$

The only nontrivial Kubota symbol we come across is

$$\kappa \left(\begin{pmatrix} 1 & & \\ & x & y \\ & -p^j & -b \end{pmatrix} \right) = \left(\frac{-p^j}{-b} \right) = \left(\frac{b}{p} \right)^j.$$

If instead we have $j = 0$ and $\min\{i, k\} > 0$, so $a = 0$ and $p \nmid c$. Then the entire matrix is built from the three multiplications

$$\begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & xp^k \end{pmatrix} \gamma_1 \begin{pmatrix} x & & y \\ & 1 & \\ -p^i & & -c \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{pmatrix} = \begin{pmatrix} p^i & & c \\ & 1 & b \\ & & p^k \end{pmatrix},$$

and the only nontrivial Kubota symbol is

$$\kappa \left(\begin{pmatrix} x & & y \\ & 1 & \\ -p^i & & -c \end{pmatrix} \right) = \left(\frac{c}{p} \right)^i.$$

We now assume $k = 0$ and $\min\{i, j\} > 0$, so $b = c = 0$ and $p \nmid a$. Then

$$\begin{pmatrix} & -1 & \\ 1 & xp^j & \\ & & 1 \end{pmatrix} \gamma_1 \begin{pmatrix} x & y & \\ -p^i & -a & \\ & & 1 \end{pmatrix} = \begin{pmatrix} p^i & a & \\ & p^j & \\ & & 1 \end{pmatrix},$$

and the Kubota symbol of the right matrix is

$$\kappa \left(\begin{pmatrix} x & y & \\ -p^i & -a & \\ & & 1 \end{pmatrix} \right) = \left(\frac{a}{p} \right)^i.$$

Finally we assume that each of i, j, and k are 6. If $i = 6$, then the only representative if γ_1 itself, which clearly has trivial Kubota symbol. If $j = 6$, then we can use

$$\begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 \end{pmatrix} \gamma_1 \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & a & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & \\ & p^6 & \\ & & 1 \end{pmatrix},$$

and again the Kubota symbol is trivial. Finally if $k = 6$, we have

$$\begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix} \gamma_1 \begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix} \begin{pmatrix} 1 & & c \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & b \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & c \\ & 1 & b \\ & & p^6 \end{pmatrix},$$

and once again the Kubota symbol is trivial.

7.4 The Whittaker coefficient

Let ϕ be a metaplectic form and write $\boldsymbol{a}(n_1, n_2)$ to refer to its (n_1, n_2)-Whittaker coefficient. This can be computed by the equation

$$\begin{aligned} &\int_{N(\mathcal{O})\backslash N(F)} \phi\left(\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}, 1\right)\tau\right) \exp(-n_1 x_1 - n_2 x_2) dx_1 dx_2 dx_3 \\ &= \boldsymbol{a}(n_1, n_2) |n_1^{-2} n_2^{-2}| W_\phi\left(\begin{pmatrix} n_1 n_2 & & \\ & n_2 & \\ & & 1 \end{pmatrix}\tau\right) \end{aligned} \tag{7.4}$$

where $N(R)$ denotes the group of elements of $\mathrm{GL}_3(R)$ of the form

$$\begin{pmatrix} 1 & a & c \\ & 1 & b \\ & & 1 \end{pmatrix}.$$

Also, $\exp : F \longrightarrow \mathbb{C}$ is a unitary additive character on F with kernel $\mathcal{O}$, and $n_1, n_2 \in \mathcal{O}$. This integral is well defined by the automorphicity of ϕ and the fact that $N(\mathcal{O})$ lifts to $\widetilde{G}$ by the map $n \longrightarrow (n, 1)$ (that is, the cocycle is trivial when restricted to $N(\mathcal{O})^2$).

Remark 7.2. We have yet to define the Whittaker function $W_\phi : \widetilde{\mathrm{GL}_3}(F) \longrightarrow \mathbb{C}$. Instead, we may simply require ϕ to be scaled so that $\boldsymbol{a}(1, 1) = 1$, and then we get that W_ϕ is defined by (7.4) as a Whittaker integral of ϕ. Also, we note that the normalization $|n_1^{-2}n_2^{-2}|$ is used to simplify the later calculations.

The strategy for determining the coefficients of our metaplectic form ϕ is as follows. We apply a Hecke operator T_g to the form, and then compute the Whittaker coefficients of $T_{\gamma_1}\phi$ by taking the above Fourier integrals. The integrals can be rewritten to be in the form of Fourier integrals of ϕ, and so we can rewrite the coefficients $T_g\phi$ as linear combinations of the coefficients of ϕ. Finally, we use the fact that the form is an eigenfunction of the Hecke algebra to rewrite the $T_g\phi$ as $\lambda\phi$, and so the linear combinations of coefficients of ϕ may be equated to the original coefficient of ϕ multiplied by the eigenvalue λ.

Now consider the Hecke operator T_{γ_1}. In order to compute the Fourier integral of $T_{\gamma_1}\phi$, we must first calculate the integrals $\phi|_\eta$ for each $\eta \in H$, the set of right coset representatives determined in section 7.7. Write $\eta \in H$ as

$$\eta = \begin{pmatrix} p^i & a & c \\ & p^j & b \\ & & p^k \end{pmatrix}.$$

Then the (n_1, n_2) Whittaker coefficient of $\phi|_\eta$ can be determined using equation 7.4. We have

$$\int_{N(\mathcal{O})\backslash N(F)} \phi\left(\begin{pmatrix} p^i & a & c \\ & p^j & b \\ & & p^k \end{pmatrix}\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\tau\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3$$

$$= \int \phi\left(\begin{pmatrix} p^i & p^ix_1 + a & p^ix_3 + ax_2 + c \\ & p^j & p^jx_2 + b \\ & & p^k \end{pmatrix}\tau\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3$$

$$= \int \phi\left(\begin{pmatrix} 1 & p^{i-j}x_1 + ap^{-j} & p^{i-k}x_3 + ax_2p^{-k} + cp^{-k} \\ & 1 & p^{j-k}x_2 + bp^{-k} \\ & & 1 \end{pmatrix}\begin{pmatrix} p^i & & \\ & p^j & \\ & & p^k \end{pmatrix}\tau\right)$$
$$\cdot \exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3$$

Now apply the substitution

$$\begin{aligned} u_1 &= p^{i-j}x_1 + ap^{-j} & u_2 &= p^{j-k}x_2 + bp^{-k} & u_3 &= p^{i-k}x_3 + ax_2p^{-k} + cp^{-k} \\ dx_1 &= p^{j-i}du_1 & dx_2 &= p^{k-j}du_2 & dx_3 &= p^{k-i}du_3 - ap^{k-i-j}du_2 \end{aligned}$$
$$dx_1dx_2dx_3 = p^{2k-2i}du_1du_2du_3.$$

The integral above is then equal to

$$\begin{aligned} &\int \phi\left(\begin{pmatrix}1 & u_1 & u_3\\ & 1 & u_2\\ & & 1\end{pmatrix}\begin{pmatrix}p^i & & \\ & p^j & \\ & & p^k\end{pmatrix}\tau\right)\exp(-n_1(p^{j-i}u_1 - ap^{-i}) - n_2(p^{k-j}u_2 - bp^{-j}))p^{2k-2i}du_i \\ &\qquad = |p^{2k-2i}|\exp(an_1p^{-i} + bn_2p^{-j})\int\phi\left(\begin{pmatrix}1 & u_1 & u_3\\ & 1 & u_2\\ & & 1\end{pmatrix}\begin{pmatrix}p^i & & \\ & p^j & \\ & & p^k\end{pmatrix}\tau\right) \\ &\qquad\qquad\cdot\exp(-n_1p^{j-i}u_1 - n_2p^{k-j}u_2)du_1du_2du_3 \\ &\qquad = |p^{2k-2i}|\exp(an_1p^{-i} + bn_2p^{-j})\boldsymbol{a}(n_1p^{j-i}, n_2p^{k-j})|(n_1p^{j-i})^{-2}(n_2p^{k-j})^{-2}| \\ &\qquad\qquad\cdot W_\phi\left(\begin{pmatrix}n_1n_2p^{k-i} & & \\ & n_2p^{k-j} & \\ & & 1\end{pmatrix}\begin{pmatrix}p^i & & \\ & p^j & \\ & & p^k\end{pmatrix}\tau\right) \\ &= \exp(an_1p^{-i} + bn_2p^{-j})\boldsymbol{a}(n_1p^{j-i}, n_2p^{k-j})|n_1^{-2}n_2^{-2}|W_\phi\left(\begin{pmatrix}p^k & & \\ & p^k & \\ & & p^k\end{pmatrix}\begin{pmatrix}n_1n_2 & & \\ & n_2 & \\ & & 1\end{pmatrix}\tau\right). \end{aligned}$$

This proves the following identity which will be used later on.

Lemma 7.3.

$$\begin{aligned} \int\limits_{N(\mathcal{O})\backslash N(F)} &\phi\left(\begin{pmatrix}p^i & a & c\\ & p^j & b\\ & & p^k\end{pmatrix}\begin{pmatrix}1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix}\tau\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3 \\ &= \exp(an_1p^{-i} + bn_2p^{-j})\boldsymbol{a}(n_1p^{j-i}, n_2p^{k-j})|n_1^{-2}n_2^{-2}| \\ &\qquad\cdot W_\phi\left(\begin{pmatrix}p^k & & \\ & p^k & \\ & & p^k\end{pmatrix}\begin{pmatrix}n_1n_2 & & \\ & n_2 & \\ & & 1\end{pmatrix}\tau\right). \end{aligned} \tag{7.5}$$

To simplify notation, we define the sixth or zeroth power of our residue symbols to be 1 identically, even when p, the modulus, divides the residue. Now we finally combine all of the calculations above. We have that

$$\int_{N(\mathcal{O})\backslash N(F)} (T_{(\gamma_1,1)}\phi)\left(\left(\left(\begin{pmatrix}1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix}, 1\right)\tau\right)\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3$$

$$= \sum_{\eta\in H}\int (\phi|_{(\eta,\zeta_\eta)})\left(\left(\left(\begin{pmatrix}1 & x_1 & x_3\\ & 1 & x_2\\ & & 1\end{pmatrix}, 1\right)\tau\right)\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3$$

$$= \sum_{\substack{i+j+k=6\\ i,j,k\geq 0}}\left[\sideset{}{'}\sum_{\substack{0\leq a<p^j\\ 0\leq b,c<p^k}}\left(\frac{a-cb'p^j}{p}\right)^i\left(\frac{b}{p}\right)^{i+j}\exp(an_1p^{-i}+bn_2p^{-j})\right]$$

$$\cdot\, \boldsymbol{a}(n_1p^{j-i}, n_2p^{k-j})|n_1^{-2}n_2^{-2}|W_\phi\left(\left(\begin{pmatrix}p^k & & \\ & p^k & \\ & & p^k\end{pmatrix}\begin{pmatrix}n_1n_2 & & \\ & n_2 & \\ & & 1\end{pmatrix}\tau\right)\right).$$

Where the prime on the inner sum means that we require the additional restrictions:

- $p\nmid a$ exactly when $\min\{i,j\}\geq 1$,
- $p\nmid b$ exactly when $\min\{j,k\}\geq 1$, and
- $p\nmid c$ exactly when $\min\{i,k\}\geq 1$ and $j=0$.

Now, we claim that in all cases, the inner sum can be written as the product of two Gauss sums. When $j\neq 0$, we have

$$p^k\cdot\left[\sum_{0\leq a<p^j}\left(\frac{a}{p}\right)^i\exp(an_1p^{-i})\right]\cdot\left[\sum_{0\leq b<p^k}\left(\frac{b}{p}\right)^{i+j}\exp(bn_2p^{-j})\right], \tag{7.6}$$

where it is assumed that $p\nmid a$ unless $i=0$, and similarly $p\nmid b$ unless $k=0$. If $j=0$, we instead have

$$p^k\cdot\left[\sum_{0\leq c<p^k}\left(\frac{c}{p}\right)^i\right],$$

since the product of b and b' cancel. In fact, this sum equals zero unless $i=0$ or $k=0$, in which case it equals p^{12} or 1, respectively.

Now, we know that the Whittaker coefficients of ϕ, $\boldsymbol{a}(n_1,n_2)$, are multiplicative, so

$$\boldsymbol{a}(n_1n_1', n_2n_2') = \boldsymbol{a}(n_1,n_2)\boldsymbol{a}(n_1',n_2')$$

when $(n_1n_2, n_1'n_2') = 1$. This implies that if the p-power coefficients (i.e. $\boldsymbol{a}(p^{\alpha_1}, p^{\alpha_2})$) are known for every p, then all of the coefficients may be calculated. Therefore, we restrict our computations to the prime power case.

So we let $n_i = p^{\alpha_i}$ for $i = 1, 2$, and $\alpha_i \geq 0$. Then for each fixed i, j, k triple in the above sum there are only certain values of (α_1, α_2) for which the Gauss sums in equation 7.6 are nonzero. The notation, and evaluation, for our Gauss sums is:

$$g_i(p^j, p^k) = \sum_{\substack{a \pmod{p^k} \\ (p \nmid a)^*}} \left(\frac{a}{p}\right)^i \exp\left(\frac{ap^j}{p^k}\right) = \begin{cases} p^j g_i(1,p) & \text{if } j+1 = k, 6 \nmid i \\ p^k & \text{if } j \geq k, 6|i \\ 0 & \text{otherwise.} \end{cases}$$

The condition in the sum that $p \nmid a$ is dropped when $6|i$.

This evaluation allows us to determine which right coset representatives contribute to which coefficient, indexed by pairs (α_1, α_2). First, assuming $j \neq 0$, we can rewrite the above equation as

$$p^k \cdot \left[g_i(p^{\alpha_1 - i + j}, p^j)\right] \cdot \left[g_{i+j}(p^{\alpha_2 - j + k}, p^k)\right].$$

Then to make the evaluations of both Gauss sums nonzero, we require either

1. $\alpha_1 = i - 1$ and $\alpha_2 = j - 1$ when $i, k > 0$,
2. $\alpha_1 = i - 1$ and $\alpha_2 \geq j$ when $i > 0$, $k = 0$,
3. $\alpha_2 = j - 1$ when $i = 0$, $k > 0$, or
4. $\alpha_2 \geq 6$ when $i = k = 0$ (and thus $j = 6$).

In the case that $j = 0$, we already noted that the only nonzero cases are when $i = 6$ (which further requires $\alpha_1 \geq 6$) or when $k = 6$, and the Gauss sums equal 1 and p^{12} in these cases respectively.

After careful computations, these restrictions and the fact that ϕ is an eigenfunction of the Hecke operator imply that the dependencies between the Whittaker coefficients $\boldsymbol{a}(p^{\alpha_i}, p^{\alpha_2})$ are in four independent 'orbits'. These orbits may be written as the sets of pairs (α_1, α_2), for $0 \leq \alpha_i \leq 5$. In this notation, we have the four orbits

$$\{(0,0),(3,0),(4,1),(4,4),(1,4),(0,3)\}, \quad \{(1,0),(2,1),(3,2),(4,3),(2,4),(0,2)\},$$
$$\{(0,1),(2,0),(4,2),(3,4),(2,3),(1,2)\}, \quad \{(1,1),(3,3)\}$$

The ratios of two coefficients in the same orbit is fixed, so we are left with four degrees of freedom for our metaplectic form ϕ (if (α_1, α_2) does not appear for $0 \le \alpha_i \le 5$, then $\boldsymbol{a}(p^{\alpha_1}, p^{\alpha_2})$ can be shown to be forced to be zero.

These results agree with Hoffstein's work ([9]). We now ask whether additional information about these dependencies can be determined by a similar analysis of the γ_2 coefficient.

7.5 Computing for γ_2

7.5.1 Right coset representatives from γ_1

Again fix a prime in $\mathfrak{O}$ and consider now the Hecke operator $T_{(\gamma_2,1)}$ where γ_2 is the matrix

$$\gamma_2 = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & 1 \end{pmatrix}.$$

Recall from the previous section that

$$\Gamma\gamma_1\Gamma = \bigcup_{\mu \in H} \Gamma\mu,$$

where H is an explicit set of right coset representatives given in equation 7.2. We now use these to provide a complete list of right coset representatives for $\Gamma\gamma_2\Gamma$. Let $w = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$. Then the set of elements

$$H' = \left\{ w \cdot \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \cdot (\mu^{-1})^T \cdot w^{-1} \right\}_{\mu \in H}$$

is a complete list of upper triangular right coset representatives for $\Gamma\gamma_2\Gamma$.

This will be a straightforward consequence of the following claim.

Proposition 7.4. *The set*

$$\left\{ \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \mu^{-1} \right\}_{\mu \in H}$$

is a complete list of left coset representatives for $\Gamma\gamma_2\Gamma$.

Proof. Since $\Gamma\gamma_1\Gamma = \bigcup_{\mu\in H}\Gamma\mu$, in particular for each $\mu \in H$, there are some $\beta,\delta \in \Gamma$ such that $\gamma_1 = \beta\mu\delta$, or equivalently,

$$\gamma_1^{-1} = \delta^{-1}\mu^{-1}\beta^{-1}.$$

Multiplying both sides by the matrix $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}$, we have the relation

$$\begin{pmatrix} 1 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \cdot \gamma_1^{-1} = \delta^{-1}\left(\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\mu^{-1}\right)\beta^{-1}$$

and thus also the relation

$$\begin{aligned}
\left(\delta\begin{pmatrix} & & 1 \\ & 1 & \\ -1 & & \end{pmatrix}\right)\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & 1 \end{pmatrix}\left(\begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}\beta\right) &= \delta\begin{pmatrix} 1 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\beta && (7.7)\\
&= \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\mu^{-1}, && (7.8)
\end{aligned}$$

which shows that $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\mu^{-1} \in \Gamma\gamma_2\Gamma$ for each $\mu \in H$. In addition, each element $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\mu^{-1}$ is in a unique left coset of $\Gamma\gamma_2\Gamma$. This follows from the fact that

$$\begin{aligned}
\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\mu^{-1}\Gamma = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}\mu'^{-1}\Gamma \quad &\iff \quad \mu^{-1}\Gamma = \mu'^{-1}\Gamma \\
&\iff \quad \Gamma\mu = \Gamma\mu' \iff \mu = \mu'.
\end{aligned}$$

Finally, we show that if $\nu\Gamma$ is a left coset of $\Gamma\gamma_2\Gamma$, there is some some $\mu \in H$ such that $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \mu^{-1}\Gamma = \nu\Gamma$. By the same method as in equation 7.7 it follows that $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \nu^{-1} \in \Gamma\gamma_1\Gamma$. By the coset decomposition from before, this implies that there is some $\mu \in H$ such that $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \nu^{-1} \in \Gamma\mu$, or equivalently,

$$\Gamma \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \nu^{-1} = \Gamma\mu.$$

Inverting and multiplying both sides by $\begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}$ then gives

$$\nu\Gamma = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \nu^{-1}\Gamma. \qquad \blacksquare$$

We have now shown that the double coset $\Gamma\gamma_2\Gamma$ can be decomposed as a union of disjoint left cosets as

$$\Gamma\gamma_2\Gamma = \bigcup_{\mu \in H} \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \mu^{-1}\Gamma.$$

To prove Theorem 7.5.1 we observe that taking the transpose of both sides swaps left and right cosets, while fixing $\Gamma\gamma_2\Gamma$. Therefore we have

$$\Gamma\gamma_2\Gamma = \bigcup_{\mu \in H} \Gamma \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} (\mu^{-1})^T.$$

7.5.2 The Kubota calculation

In this section we show how we can obtain the appropriate Kubota symbol attached to each of the coset representatives in H'. Recall that each element of H' is of the form $w \cdot \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix} \cdot (\mu^{-1})^T \cdot w^{-1}$ where each $\mu \in H$. Recall also the following notation:

$$\gamma_1 = \begin{pmatrix} p^6 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & 1 \end{pmatrix}, \ \gamma = \begin{pmatrix} p^6 & & \\ & p^6 & \\ & & p^6 \end{pmatrix}.$$

In the previous section, we provided for each $\mu \in H$ matrices $\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_m \in \Gamma$ such that

$$\alpha_1 \cdot ... \cdot \alpha_k \cdot \gamma_1 \cdot \beta_1 \cdot ... \cdot \beta_m = \mu \tag{7.9}$$

and such that their Kubota symbols were easy to compute. For short, we denote the products of these matrices by

$$\overrightarrow{\alpha} = \alpha_1 \cdot ... \cdot \alpha_k$$

$$\overrightarrow{\beta} = \beta_1 \cdot ... \cdot \beta_m.$$

Inverting both sides of equation 7.9 we have

$$\overrightarrow{\beta}^{-1} \gamma_1^{-1} \overrightarrow{\alpha}^{-1} = \mu^{-1},$$

and multiplying by γ then gives

$$\overrightarrow{\beta}^{-1} \left(\gamma\gamma_1^{-1}\right) \overrightarrow{\alpha}^{-1} = \gamma\mu^{-1}.$$

Let $w = \begin{pmatrix} & & -1 \\ & 1 & \\ 1 & & \end{pmatrix}$, then

$$\gamma_2 = w^{-1}\gamma\gamma_1 w$$

and so we have the relation

$$\overrightarrow{\beta}^{-1} w\gamma_2 w^{-1} \overrightarrow{\alpha}^{-1} = \gamma\mu^{-1}.$$

Taking the transpose then gives

$$(\overrightarrow{\alpha}^{-1})^T w\gamma_2 w^{-1}(\overrightarrow{\beta}^{-1})^T = \gamma(\mu^{-1})^T.$$

Finally, conjugate both sides by w:

$$w(\overrightarrow{\alpha}^{-1})^T w\gamma_2 w^{-1}(\overrightarrow{\beta}^{-1})^T w^{-1} = w\gamma(\mu^{-1})^T w^{-1}.$$

On the right hand side, we have a right coset representative for the double coset $\Gamma\gamma_2\Gamma$. On the left hand side, we have an expression for this representative as a product of known matrices with simple to compute Kubota symbols. in fact, since taking the transpose or inverting a matrix inverts the Kubota symbol, and since w has trivial Kubota symbol, the result is the same as in the previous section.

7.6 Whittaker

Now we compute the Whittaker coefficients of the metaplectic form $T_{\gamma_2}\phi$ using the computations for T_{γ_1}. We have:

$$\int_{N(\mathcal{O})\backslash N(F)} (T_{(\gamma_2,1)}\phi)\left(\left(\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}, 1\right)\tau\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3\right.$$
$$= \sum_{\eta'\in H'} \int (\phi|_{(\eta',\zeta_{\eta'})})\left(\left(\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}, 1\right)\tau\right)\right)\exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3$$
$$= \sum_{\eta\in H} \int (\phi|_{(w\cdot\gamma\cdot(\eta^{-1})^T\cdot w^{-1},\zeta_\eta)})\left(\left(\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}, 1\right)\tau\right)\right)\exp(-\text{''}-)dx_1dx_2dx_3$$
$$= \sum_{\substack{i+j+k=6 \\ i,j,k\geq 0}} \left[\sum_{\substack{0\leq a<p^j \\ 0\leq b,c<p^k}}{}' \left(\frac{a - cb'p^j}{p}\right)^i \left(\frac{b}{p}\right)^{i+j}\right.$$
$$\cdot \int_{N(\mathcal{O})\backslash N(F)} \phi\left(\left(\left(\begin{pmatrix} p^{i+j} & bp^i & -(ab-cp^j) \\ & p^{i+k} & -ap^k \\ & & p^{j+k} \end{pmatrix}, 1\right)\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}, 1\right)\tau\right)\right)$$
$$\left.\cdot \exp(-n_1x_1 - n_2x_2)dx_1dx_2dx_3\right].$$

Using equation 7.5 the integral is equal to

$$\exp(bp^in_1p^{-i-j} - ap^kn_2p^{-i-k})\boldsymbol{a}(n_1p^{(i+k)-(i+j)}, n_2p^{(j+k)-(i+k)})|n_1^{-2}n_2^{-2}|$$
$$\cdot W_\phi\left(\left(\begin{pmatrix} p^{j+k} & & \\ & p^{j+k} & \\ & & p^{j+k} \end{pmatrix}\begin{pmatrix} n_1n_2 & & \\ & n_2 & \\ & & 1 \end{pmatrix}\tau\right)\right).$$

We can substitute this into the work above to express the Whittaker coefficient as

$$\sum_{\substack{i+j+k=6\\ i,j,k\geq 0}} \left[\sideset{}{'}\sum_{\substack{0\leq a<p^j\\ 0\leq b,c<p^k}} \left(\frac{a-cb'p^j}{p}\right)^i \left(\frac{b}{p}\right)^{i+j} \exp(bn_1p^{-j}-an_2p^{-i}) \right]$$
$$\cdot\, \boldsymbol{a}(n_1p^{k-j}, n_2p^{j-i})|n_1^{-2}n_2^{-2}|W_\phi \left(\begin{pmatrix} p^{j+k} & & \\ & p^{j+k} & \\ & & p^{j+k} \end{pmatrix} \begin{pmatrix} n_1n_2 & & \\ & n_2 & \\ & & 1 \end{pmatrix} \tau \right).$$

The result above is identical to the result for γ_1, except that n_1 and n_2 are switched, and the arguments of $\boldsymbol{a}(-,-)$ are swapped.

So we have answered the question posed: we do not end up with any new information about the Whittaker coefficients of ϕ by using $T_{(\gamma_2,1)}$. The explanation is that the choice between γ_1 and γ_2 is equivalent to a choice of ordering of the two simple roots of SL_3. This result would be expected for other metaplectic covers, n, but the worry is that the complexity of calculation would increase with n using the methods above. Additional interest concerns higher rank groups: with more generators for their Hecke algebras but also more roots, how many independent relations can be determined on their Whittaker coefficients?

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