

Stable splittings of surface mapping spaces[☆]

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Received 26 June 2005; received in revised form 12 December 2005; accepted 14 December 2005

Abstract

We study the homotopy type of mapping spaces from Riemann surfaces to spheres. Our main result is a stable splitting of these spaces into a bouquet of new finite spectra. From this and classical results, one may deduce splittings of the configuration spaces of surfaces.

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MSC: 55P42; 55P15; 55P35; 55R80; 55S10

Keywords: Function spaces; Surfaces; Stable splitting; Brown–Gitler spectrum

1. Introduction

Mapping spaces with Riemann surfaces as domains have been an object of much recent interest. In physics, such spaces describe the worldsheets of a string evolving through time. In algebraic and symplectic geometry, one puts algebraic or analytic restrictions on the types of maps allowed, and studies the resulting moduli space; this is Gromov–Witten theory. When we put no requirements on these functions other than continuity, remarkable results in stable homotopy theory have been obtained.

Strikingly, in homotopy theory as in the other theories, much less is known for Riemann surfaces with genus greater than 0 than in the case of the Riemann sphere. Much of what we do know is comprised of several works. In [3], Bödigheimer, F. Cohen, and Taylor compute the ranks of the cohomology groups of these mapping spaces when the target is a sphere (as well as a host of other mapping spaces with different domains). F. Cohen, R. Cohen, Mann and Milgram prove a periodicity of their stable homotopy type as the dimension of the target sphere varies in [8]. Finally, in [2], Bödigheimer, F. Cohen and Milgram construct some stable and unstable splittings of these spaces, particularly at primes greater than 3. Missing from these calculations is a full understanding of the cohomology operations on these spaces, an essential part of the homotopy theory of these function spaces. On the other hand, this information was known for genus 0 as early as 1976 in F. Cohen’s work in [9].

This is the main subject of the present paper: We study the stable homotopy type of the function spaces $\text{Map}_*(X_g, S^n)$ of continuous based maps from a compact, orientable surface X_g of genus g without boundary to a sphere of dimension n . That type determines all generalized cohomology information about the space, such as

[☆] This material is based upon work supported in part by the National Science Foundation under agreement No. DMS-0111298.

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K -theory, cobordism groups, and cohomology operations. In many cases we will split these function spaces into recognizable finite complexes; in the remaining cases we will state some conjectures on extending these results.

The genus 0 case (the spaces $\Omega^2 S^n$) has been studied by many authors, including [1,6,9,10,13,16], evidencing the great interest of homotopy theorists in the theory of loop spaces. There have been some spectacular results, including Mahowald’s construction of an infinite family of elements in the stable homotopy groups of spheres detected at filtration 2 in the Adams spectral sequence (thus disproving the so-called Doomsday Conjecture in stable homotopy theory) in [16], and the realization in [16,12] of the Eilenberg–MacLane spectra $H\mathbb{Z}$, and $H\mathbb{Z}/2$ as Thom spectra, amongst others. Thus, we will focus on the genus > 0 case.

We will restrict our focus to 2-primary information regarding the stable homotopy types of the spaces $\text{Map}_*(X_g, S^n)$, i.e., unless we indicate otherwise, coefficients for all homology and cohomology theories are taken to be the field \mathbb{F}_2 of two elements. We focus on the prime 2 because rationally and for odd primes, odd dimensional spheres are H -spaces; thus a collapse like Proposition 1.1 below occurs. Even-dimensional spheres are more difficult; away from 2, an important spectral sequence fails to collapse (see [2, Theorems 10.2 and 11.4]).

Let us now be more specific. \mathcal{A}^* will denote the Steenrod algebra of cohomology operations for $H^*(\cdot; \mathbb{F}_2)$. In [4], Brown and Gitler introduced the cyclic module

$$M(k) := \mathcal{A}^*/\mathcal{A}^*\{\chi(Sq^i) \mid i > k\}$$

(here $\chi(Sq^i)$ denotes the conjugate of Sq^i in \mathcal{A}^*), and a 2-local spectrum $B(k)$ whose mod 2 cohomology is $M(k)$.

Snaith’s stable splitting [21] of $\Omega^k \Sigma^k X$ was utilized by Mahowald to construct his family $\eta_i \in \pi_{2i}^S$. For $n > 2$, the stable wedge summands of $\Omega^2 S^n$ are equivalent to Thom spaces of certain vector bundles over the configuration spaces of \mathbb{R}^2 . Mahowald identified the cohomology of these summands as that of suspensions of Brown–Gitler spectra (for n odd). He conjectured that at the prime 2, they were in fact Brown–Gitler spectra, and in [6], Brown and Peterson proved this to be true. They also extend the results to the configuration spaces themselves, as well as the case of n even.

To extend these results to a stable splitting of $\text{Map}_*(X_g, S^n)$ for $g > 0$, a good place to begin is with $n = 3$. The following was noted in [2] and, employing the group structure in S^3 , is not difficult to show:

Proposition 1.1. *The function space of maps from X_g to S^3 , $\text{Map}_*(X_g, S^3)$ is homotopy equivalent to $\Omega^2 S^3 \times (\Omega S^3)^{\times 2g}$.*

Since ΩS^3 is stably equivalent to a wedge of spheres, it follows that $\text{Map}_*(X_g, S^3)$ stably decomposes into a wedge of suspensions of Brown–Gitler spectra. A result of [8] implies that (for $n \geq 2$) the stable homotopy types of the Snaith summands of $\text{Map}_*(X_g, S^n)$ are 4-periodic in n , up to suspension. Thus the stable homotopy type of $\text{Map}_*(X_g, S^n)$ is understood for $n = 3 \pmod 4$ and all g .

We focus, then, on $n \not\equiv 3 \pmod 4$. In these cases, new spectra are needed. They are certain elaborations of the Brown–Gitler spectra in the presence of the mod 2 Moore spectrum $M\mathbb{Z}/2$ and the cofiber $C\eta$ of the Hopf fibration $\eta: S^1 \rightarrow S^0$. Let $X^{\wedge n}$ denote the n -fold smash power of X .

Theorem 1.2. *For each $g, k \geq 0$, there exist spectra $N_g(k)$, $P_g(k)$, and $L_g(k)$.*

(1) $H^*(N_g(k))$ is the quotient of $\Sigma^k M \lfloor \frac{k}{2} \rfloor \otimes H^*(M\mathbb{Z}/2^{\wedge g})$ by the \mathcal{A}^* -submodule generated by the set $\{\chi(Sq^t) \otimes x \mid \dim x = n, t > \lfloor \frac{k-n}{2} \rfloor\}$ (ignoring the suspension coordinate). Moreover, this quotient is realized by a map

$$N_g(k) \rightarrow \Sigma^k B \left\lfloor \frac{k}{2} \right\rfloor \wedge (M\mathbb{Z}/2)^{\wedge g}.$$

(2) $H^*(P_g(k))$ is the quotient of $\Sigma^{3k} M \lfloor \frac{k}{2} \rfloor \otimes H^*(C\eta^{\wedge g})$ by the \mathcal{A}^* -submodule generated by the set $\{\chi(Sq^t) \otimes x \mid \dim x = n, t > \lfloor \frac{k-n}{2} \rfloor\}$ (ignoring the suspension coordinate). Moreover, this quotient is realized by a map

$$P_g(k) \rightarrow \Sigma^{3k} B \left\lfloor \frac{k}{2} \right\rfloor \wedge (C\eta)^{\wedge g}.$$

(3) $H^*(L_g(k))$ is a quotient of

$$\Sigma^{4k} \left(\bigoplus_{i=0}^k H^*(N_g(i)) \right) \otimes H^*(C\eta^{\wedge g}). \tag{*}$$

Write $H^*(M\mathbb{Z}/2^{\wedge g})$ and $H^*(C\eta^{\wedge g})$ as exterior algebras on g generators x_1, \dots, x_g of dimension 1, and y_1, \dots, y_g of dimension 2 respectively. So if we write $H^*(N_g(i))$ as a quotient of $M[\frac{i}{2}] \otimes \Lambda[x_1, \dots, x_g] \cdot e_i$ (where e_i is a generator of dimension i), $H^*(L_g(k))$ is the quotient of (*) by the \mathcal{A}^* -submodule generated by

$$y_{j_1} \cdots y_{j_n} e_{i-n} - x_{j_1} \cdots x_{j_n} e_i.$$

So, for instance, the cohomology of $P_g(k)$ is the same as the cohomology of the $3k$ suspension of

$$\bigvee_{j=0}^g \Sigma^{2j} B \left\lfloor \frac{k-2j}{2} \right\rfloor$$

but with cohomology operations perturbed by the presence of the $C\eta$'s. Similarly, the description of $H^*(L_g(k))$ implies that it is isomorphic to $\Sigma^{4k} \bigoplus_{i=0}^k H^*(N_g(i))$ as vector spaces, as all of the new elements (multiples of y_i) are identified with elements of $\Sigma^{4k} \bigoplus_{i=0}^k H^*(N_g(i))$. However, the Steenrod operations on $H^*(L_g(k))$ are different; as constructed, there are operations which connect the i th copy of $H^*(N_g(i))$ with the $(i + 1)$ st.

The construction of these spectra employs the stable decomposition of $\Omega^2 S^n$ and its structure as a ring spectrum. More specifically, they are cofibers of maps defined between the stable summands of $\Omega^2 S^n$ using the loop multiplication on that space. Specific details of their construction occur in Section 6.

One can easily see from its construction that $N_1(2k)$ splits into a wedge of $\Sigma^{2k} B(k)$ and $\Sigma^{2k+1} B(k - 1)$, and that $N_1(2k + 1) = \Sigma^{2k+1} B(k) \wedge M\mathbb{Z}/2$. We are lead to conjecture similar splittings of $N_g(k)$ for all g , but as yet do not have a proof.

Consider $\text{Map}_*(X_g, S^2)$. This space decomposes as a union of (homotopy equivalent) components, given by the degree of the map. Take $\text{Map}_*(X_g, S^2)_n$ to be the component of maps of degree n . Let X_+ denote the addition of a disjoint basepoint to a space X , and let $X^{\vee n}$ denote the n -fold wedge of X .

Theorem 1.3. $\text{Map}_*(X_g, S^2)_{0+}$ is stably equivalent to

$$\left(\bigvee_{i=0}^g \left(\Sigma^i \bigvee_{k=0}^{\infty} N_{g-i}(k) \right)^{\vee 2^i \binom{g}{i}} \right) \wedge \left(\bigvee_{j=0}^{\infty} S^{2j} \right)^{\wedge 2g}. \tag{**}$$

One can use Theorem 1.3 to give a complete description of the stable homotopy type of the configuration spaces of punctured surfaces. We give this splitting explicitly in Section 7.2.

Let us describe how the theorem is proved. The 1-skeleton of X_g is a wedge of $2g$ circles, $(S^1)^{\vee 2g}$. Restriction of maps from X_g to $(S^1)^{\vee 2g}$ gives a fibration

$$\text{Map}_*(X_g, S^n) \rightarrow (\Omega S^n)^{\times 2g}$$

with fibre $\Omega^2 S^n$. In [2] Bödiger et al. define $Y_{g,n}$ to be the pullback of this fibration over the map

$$(S^{n-1})^{\times 2g} \xrightarrow{E^{\times 2g}} (\Omega S^n)^{\times 2g}$$

where $E : S^{n-1} \rightarrow \Omega S^n$ is the suspension map (in the case $n = 2$, we define $Y_{g,2}$ to be the restriction of this pullback to $\text{Map}_*(X_g, S^2)_0$). A result of [2] is that $\text{Map}_*(X_g, S^2)_0 \simeq Y_{g,2} \times (\Omega S^3)^{\times 2g}$. It is well known that $\Omega S^3_+ \simeq \bigvee_{j \geq 0} S^{2j}$, so to prove Theorem 1.3, we must split $Y_{g,2+}$ into the first term of (**); this is done in Section 6.

The equivalence $\text{Map}_*(X_g, S^2)_0 \simeq Y_{g,2} \times (\Omega S^3)^{\times 2g}$ is proven using the Hopf–James fibration $\Omega S^{2n} \rightarrow \Omega S^{4n-1}$ (with fiber S^{2n-1} , given by E) and the action of S^3 on S^2 . Consequently, the same methods do not apply in an attempt to do the same for $\text{Map}_*(X_g, S^n)$ with $n > 2$ (unless, as in [2], one inverts 6 to give S^{4n-1} an action on S^{2n} . As we are concerned solely with 2-primary information in this paper, this avenue is inappropriate here). However, we do have the following result.

Proposition 1.4. *As a module over \mathcal{A}^* ,*

$$H^*(\text{Map}_*(X_g, S^n)) \cong H^*(Y_{g,n}) \otimes H^*(\Omega S^{2n-1})^{\otimes 2g}.$$

Recall that the Steenrod operations on $H^*(\Omega S^m)$ are trivial for any m , since ΩS^m is stably equivalent to $\bigvee_{k \geq 1} S^{k(m-1)}$. On the basis of this cohomological evidence, we make the following conjecture:

Conjecture 1.5. *$\text{Map}_*(X_g, S^n)$ is stably equivalent to $Y_{g,n} \times (\Omega S^{2n-1})^{\times 2g}$.*

Currently we do not have a proof of this conjecture. We can, however, stably split $Y_{g,n+}$ in a fashion similar to the splitting of $\text{Map}_*(X_g, S^2)_0$:

Definition 1.6. Let $M_{g,2}(2k) = N_g(k)$ and $M_{g,2}(2k + 1)$ be contractible. Let $M_{g,4}(2k) = L_g(k)$ and $M_{g,4}(2k + 1) = \Sigma^2 M_{g,4}(2k)$. Set $M_{g,5}(k) = P_g(k)$. Define $M_{g,6}(k) = \Sigma^{4k} \bigvee_{i=0}^k M_{g,2}(i)$, and for $n > 2$ and $n \not\equiv 3 \pmod{4}$, set $M_{g,n+4}(k) = \Sigma^{4k} M_{g,n}(k)$.

Theorem 1.7. *For $n \geq 2$ and $n \not\equiv 3 \pmod{4}$, $Y_{g,n+}$ is stably equivalent to*

$$\bigvee_{i=0}^g \left(\Sigma^{(n-1)i} \bigvee_{k=0}^{\infty} M_{g-i,n}(k) \right)^{\vee 2^i \binom{g}{i}}.$$

We ask the reader’s patience with a splitting in which some of the summands (in the case $n = 2 \pmod{4}$) are contractible. Defining $M_{g,2}(k)$ as we have allows the coordination of the index k with a certain filtration on $\text{Map}_*(X_g, S^2)_0$ defined in [18]. We refer the reader to Proposition 6.2 for more details.

The spectra $M_{g,n}(k)$ are easy enough to construct; to show that they form the stable summands of the mapping space is more difficult. To do this, we need a computation of the Steenrod operations on the mapping spaces. Stably, these spaces are (wedges of) Thom spaces of bundles over certain configuration spaces. Thus to understand the Steenrod operations we need to know the cohomology operations on the base, the Stiefel–Whitney classes of the bundles, and how they interact—the ring structure of the cohomology of the configuration spaces. This information is computed in Section 4 and assembled into a splitting in Section 6, using some preliminary work in Section 5.

In Section 2 we summarize some necessary background, and in Section 3 introduce a multiplicative structure which is used countless times in the proofs. Section 7 consists largely of the construction of a certain filtration on the homology of $\text{Map}_*(X_g, S^2)_0$. It is used in 7.3 to give a proof of Proposition 1.4. Finally, in Section 8, we make some remarks towards extending these results to spaces of unbased maps. Cohomology groups are computed, and the natural splitting one is led to conjecture is shown to be naive.

2. Background

2.1. Configuration models of mapping spaces

A theory of approximations for mapping spaces has been developed in numerous papers, including [17–19,21]; a good cumulative reference is [5]. There are, for sufficiently nice spaces X and Y , combinatorially defined models for the mapping space $\text{Map}(X, Y)$. These models are defined quite similarly to the models for $\Omega^n \Sigma^n Y$ constructed using the little n -cubes operad in [17]. Snaith, in [21], takes advantage of an evident filtration on these models to provide stable splittings of these mapping spaces into wedges of filtration quotients of their models. In this section we give the briefest of introductions to this theory.

Recall that for a space X , the k th ordered configuration space of X is the space of k distinct labelled points in X , defined as $\tilde{C}^k(X) = X^k \setminus \Delta$, where $\Delta = \{(x_1, \dots, x_k), \exists i \neq j, x_i = x_j\}$ is the “fat diagonal”. There is an action of the k th symmetric group, Σ_k , on this space by permutation of coordinates. Notice that by removing Δ , we have removed the fixed points of this action. The quotient $C^k(X) = \tilde{C}^k(X)/\Sigma_k$ is called the k th (unordered) configuration space of X and the map $\tilde{C}^k(X) \rightarrow C^k(X)$ is a covering map. We will use the notation $x_1 + \dots + x_k$ for the equivalence class of (x_1, \dots, x_k) in $C^k(X)$. Also, let $\gamma_k = \tilde{C}^k(X) \times_{\Sigma_k} \mathbb{R}^k$ be the “permutation bundle” on $C^k(X)$, where the action of Σ_k on \mathbb{R}^k is by permutation of coordinates.

Let X and Y be spaces, and suppose Y has a basepoint $*$. Define the space $C(X; Y)$ to be the quotient of $\coprod_{k \geq 0} \tilde{C}(X, k) \times_{\Sigma_k} Y^k$ via the identifications:

$$(x_1, \dots, x_k; y_1, \dots, y_{k-1}, *) \sim (x_1, \dots, x_{k-1}; y_1, \dots, y_{k-1}).$$

The following theorem may be stated in more generality, but for our purposes this will suffice, since we are only concerned with models for based mapping spaces. Its proof may be found in [5], and is an elaboration of ideas in [18] and [17].

Theorem 2.1. *Let X be a compact n -manifold such that the punctured space $X \setminus \{*\}$ is parallelizable, and let Y be a path-connected space. Then there is a map $r : C(X \setminus \{*\}; Y) \rightarrow \text{Map}_*(X, \Sigma^n Y)$ which is a weak equivalence.*

The equivalence r should be thought of as sending a labelled configuration $(x_1, \dots, x_k; y_1, \dots, y_k)$ to the map from X to $\Sigma^n Y$ which wraps a small n -disc in X centered at x_i around the n -sphere $\Sigma^n(* \coprod y_i)$ in $\Sigma^n Y$ and collapses the complement of these discs to the basepoint $*$.

We notice that this theorem agrees with the approximation theorem in [17] by taking $X = S^n$; the resulting configuration model employs the (ordered) configuration spaces of $\mathbb{R}^n = S^n \setminus \{*\}$, which are Σ_k -equivariantly homotopy equivalent to the spaces of the little n -cubes operad.

There is a natural filtration on $C(X; Y)$ by the length of the configuration: let F_n be the image of $\coprod_{k=0}^n \tilde{C}(X, k) \times_{\Sigma_k} Y^k$ in $C(X; Y)$. The following theorem was proved originally in [21] for the case of the little n -cubes operad and $X = S^n$. The generalization to arbitrary manifolds X is straightforward, and may be found in [5].

Theorem 2.2 (The Snaith Splitting). *For any manifold X and CW-complex Y , there is a stable equivalence*

$$C(X; Y) \rightarrow \bigvee_{k \geq 0} F_k / F_{k-1}.$$

For our purposes, we will always take X to be a closed orientable surface and $Y = S^n$, $n > 0$ (the case $n = 0$ is discussed below). Since all punctured orientable surfaces are parallelizable, and Y is path-connected, the models described above do work. One can see (noted, for instance, in [7]) that for such Y , the filtration quotient F_k / F_{k-1} is the Thom space of the n -fold sum of the permutation bundle over $C^k(X \setminus \{*\})$. Consequently, for a closed, orientable surface X and $n > 0$, the space $\text{Map}_*(X, S^{n+2})$ stably splits as a wedge of Thom spaces

$$\text{Map}_*(X, S^{n+2}) \simeq \bigvee_{k \geq 0} C^k(X \setminus \{*\})^{n\gamma_k}.$$

Notice that for a compact, orientable surface of genus g , X_g , the map $r : C(X_g \setminus \{*\}; S^n) \rightarrow \text{Map}_*(X_g, S^{n+2})$ is only an equivalence if $n > 2$. When the dimension n of the target sphere is the same as the dimension of the domain, X , $C(X \setminus \{*\}; S^0)$ decomposes into a union of configuration spaces. And though r fails to be an equivalence, we still have maps

$$r_k : C^k(X \setminus \{*\}) \rightarrow \text{Map}_*(X, S^n).$$

In fact r_k carries $C^k(X \setminus \{*\})$ into the component $\text{Map}_*(X, S^n)_k$ of $\text{Map}_*(X, S^n)$ consisting of maps of degree k .

We may retract $X \setminus \{*\}$ away from its puncture via a retraction isotopic to the identity. Subsequently adding a point to a configuration in $C^k(X \setminus \{*\})$ near the puncture defines a map $i_k : C^k(X \setminus \{*\}) \rightarrow C^{k+1}(X \setminus \{*\})$. The maps r_k tend to a homology isomorphism as k tends to infinity over the maps i_k :

Theorem 2.3. [18] *Let X be a closed n -manifold with $X \setminus \{*\}$ parallelizable. The maps $C^k(X \setminus \{*\}) \rightarrow \text{Map}_*(X, S^n)_k$ induce an isomorphism*

$$\lim_{k \rightarrow \infty} H_*(C^k(X \setminus \{*\}); \mathbb{Z}) \cong \lim_{k \rightarrow \infty} H_*(\text{Map}_*(X, S^n)_k; \mathbb{Z}) \cong H_*(\text{Map}_*(X, S^n)_0; \mathbb{Z}).$$

If we let $C^\infty(X \setminus \{*\})$ be the homotopy limit of the maps $C^k(X \setminus \{*\}) \rightarrow C^{k+1}(X \setminus \{*\})$, then McDuff’s theorem can be interpreted as saying that the induced map $C^\infty(X \setminus \{*\}) \rightarrow \text{Map}_*(X, S^n)_0$ is a homology isomorphism and hence a stable equivalence.

These models admit a stable splitting as well; if we define

$$D^k(X \setminus \{*\}) := \text{cofiber}(C^{k-1}(X \setminus \{*\}) \rightarrow C^k(X \setminus \{*\})).$$

Then it follows from work of Cohen et al. [11] that there is a stable splitting

$$\text{Map}_*(X, S^n)_0 \simeq C^\infty(X \setminus \{*\}) \simeq \bigvee_{k>0} D^k(X \setminus \{*\}).$$

This restricts to splittings of the finite configuration spaces.

Results of F. Cohen, Mahowald, and Milgram in [10] and F. Cohen, R. Cohen, Mann, and Milgram in [8] give the order of the permutation bundles over the configuration spaces of surfaces, thus giving a periodicity theorem for the homotopy type of the Snaith summands of $\text{Map}_*(X_g, S^n)$:

Theorem 2.4. [10,8] $2\gamma_k$ is a trivial bundle over $C^k(\mathbb{R}^2)$ [10], and $4\gamma_k$ is a trivial bundle over $C^k(X_g \setminus \{*\})$ [8].

2.2. Cohomology of mapping spaces

The cohomology of $\text{Map}_*(X_g, S^n)$ (and a wealth of other mapping spaces whose codomain is a sphere) has been determined in [3] en route to determining the homology of configuration spaces. The restriction of maps from X_g to S^n to the 1-skeleton of X_g , a wedge of $2g$ circles $S^1 \vee \dots \vee S^1$, induces a fibration $r_n(g) : \text{Map}_*(X_g, S^n) \rightarrow (\Omega S^n)^{\times 2g}$ with fibre $\Omega^2 S^n$. In [3] it is shown using the configuration model of this mapping space that this fibration gives

Proposition 2.5. [3] For $n > 2$, as a vector space over \mathbb{F}_2 ,

$$H^*(\text{Map}_*(X_g, S^n)) \cong H^*(\Omega^2 S^n) \otimes H^*(\Omega S^n)^{\otimes 2g}.$$

Proposition 2.6. For a group G , $\text{Map}_*(X_g, G)$ is homotopy equivalent to $\Omega^2 G \times (\Omega G)^{\times 2g}$.

Proof. G admits a classifying space BG , and $G \simeq \Omega BG$. So $\text{Map}_*(X_g, G) \simeq \text{Map}_*(\Sigma X_g, BG)$. But $\Sigma X_g \simeq S^3 \vee (S^2)^{\vee 2g}$, so

$$\text{Map}_*(X_g, G) \simeq \Omega^3 BG \times (\Omega^2 BG)^{\times 2g} \simeq \Omega^2 G \times (\Omega G)^{\times 2g}. \quad \square$$

Notice that we obtain Proposition 1.1 as a corollary to this result.

2.3. Homology of $\Omega^2 S^3$ and Brown–Gitler spectra

In this section we collect known facts about $H_*(\Omega^2 S^3)$ as a Hopf algebra and comodule over the dual of the Steenrod algebra.

The Brown–Gitler spectra $B(n)$ are defined in [4]. $H^*(B(n))$ is the module $M(n) := \mathcal{A}^*/\mathcal{A}^*\{\chi(Sq^k), k > n\}$. Here $\chi(Sq^k)$ denotes the conjugate of Sq^k in \mathcal{A}^* .

It is well known from [9] that, if we write a_1 for the image of the fundamental class $[S^1]$ in $H_1(\Omega^2 S^3)$ under the map $S^1 \rightarrow \Omega^2 \Sigma^2 S^1$, then

$$H_*(\Omega^2 S^3) = \mathbb{F}_2[a_n, n \geq 1]$$

where $a_n = Q_1(a_{n-1})$ and Q_1 is the first (and in this case, only) Kudo–Araki–Dyer–Lashof operation. Note that $|a_n| = 2^n - 1$. Also note that we may write $a_n = Q_1^{n-1}(a_1) = Q^{2^{n-1}} \dots Q^4 Q^2 a_1$. A computation with the diagonal Cartan formula for Dyer–Lashof operations shows that a_n is primitive. We summarize some known facts about $H^*(\Omega^2 S^3)$:

Theorem 2.7.

- (1) $H_*(\Omega^2 S^3) = \mathbb{F}_2[a_n, n \geq 1]$ is a connected, primitively generated, commutative, cocommutative Hopf algebra of finite type [9].

- (2) The Snaith splitting is a stable equivalence $\Omega^2 S^3 \simeq \bigvee_{k \geq 0} F_k/F_{k-1}$; here F_k/F_{k-1} is 2-equivalent to $\Sigma^k B\lfloor \frac{k}{2} \rfloor$. Alternatively (as per the comments after Theorem 2.2), one may describe F_k/F_{k-1} as the Thom space of the permutation bundle γ_k on $C^k(\mathbb{R}^2)$ [16,1].
- (3) The space of primitives in $H_*(\Omega^2 S^3)$ is the span of $\{a_n^{2^k}; n \geq 1, k \geq 0\}$. Since $H_*(\Omega^2 S^3)$ is primitively generated, one can conclude that its dual is an exterior algebra on generators $b_{n,k}$ of dimension $2^k(2^n - 1)$ dual to $a_n^{2^k}$ in the monomial basis of $H^*(\Omega^2 S^3)$.
- (4) In [9] or [3] a weight filtration is put on the $H_*(\Omega^2 S^3)$ assigning $w(a_n) = 2^{n-1}$, and extending via $w(xy) = w(x) + w(y)$. The subspace of filtration k is the homology of the stable summand F_k/F_{k-1} . Write $k = 2^{k_1} + \dots + 2^{k_n}$ as the binary expansion. If we define a weight filtration on $H^*(\Omega^2 S^3)$ by duality, then we see that $\bar{w}_k := b_{1,k_1} \cdots b_{1,k_n}$ (dual to a_1^k) is the unique class of filtration k and dimension k . It is therefore the Thom class of γ_k .

The reason for the notation \bar{w}_k will be explained in Section 4.3.

3. Multiplicative properties

For any space Y , the n th loop space, $\Omega^n Y$, is the prototypical example of an H -space via the loop product. To be more picturesque, the collapse of the equator of S^n gives a map $S^n \rightarrow S^n \vee S^n$ which induces the product $\Omega^n Y \times \Omega^n Y \rightarrow \Omega^n Y$.

One can do similarly for maps from closed orientable surfaces if one is willing to consider all surfaces simultaneously. Consider maps (for each $g, h \geq 0$)

$$k : X_{g+h} \rightarrow X_g \vee X_h$$

which collapse a circle in X_{g+h} which separates the surface into $X_g \setminus \text{disk}$ and $X_h \setminus \text{disk}$. This induces a map

$$k^* : \text{Map}_*(X_g, Y) \times \text{Map}_*(X_h, Y) \rightarrow \text{Map}_*(X_{g+h}, Y).$$

Taken together over all choices of g and h , this defines an H -space structure on the union

$$\coprod_{g \geq 0} \text{Map}_*(X_g, Y).$$

One may choose the maps k appropriately to ensure that this is a homotopy associative product. It is natural at this point to ask what sort of algebraic structure (e.g., operad) governs this product. Such a gadget should at least encode the way that separating circles lie inside X_{g+h} . Though we will make a few remarks, it is beyond the scope of this article to explore this question in detail; for the splittings we will need only the product, not the finer structure.

We note that $\coprod \text{Map}_*(X_g, Y)$ contains $\Omega^2 Y$ as a sub- H -space. One might therefore expect some sort of action of the little disks operad on these spaces. However, we will show in Corollary 4.3 that in homology the product is not generally commutative (in particular when $Y = S^n$). Hence the little disks action must not extend. It is clear, however, that the construction makes $\coprod \text{Map}_*(X_g, Y)$ a module over $\Omega^2 Y$; in fact, it is a module over $\Omega^2 Y$ (in the operadic sense) as an algebra over the two-dimensional little disks operad.

It is easy to show that the product on $\coprod \text{Map}_*(X_g, S^n)$ restricts to a product on $\coprod Y_{g,n}$. We can also see this product geometrically in the configuration models; define a map

$$\kappa : C^i(X_g \setminus \text{disk}) \times C^j(X_h \setminus \text{disk}) \rightarrow C^{i+j}(X_{g+h} \setminus \text{disk})$$

as follows: think of $X_{g+h} \setminus \text{disk}$ as $X_g \setminus \text{disk}$ and $X_h \setminus \text{disk}$ glued along two boundaries of a pair of pants. Then configurations of i points in $X_g \setminus \text{disk}$ and j points in $X_h \setminus \text{disk}$ give the desired configuration in $X_{g+h} \setminus \text{disk}$. Since $C^k(X_g \setminus \{*\}) \simeq C^k(X_g \setminus \text{disk})$ this gives rise to an H -space structure on

$$\coprod_{g,k \geq 0} C^k(X_g \setminus \{*\})$$

which in turn induces one on the configuration models for $\coprod \text{Map}_*(X_g, \Sigma^2 Y)$. In summary:

Proposition 3.1. *The spaces $\coprod \text{Map}_*(X_g, Y)$, $\coprod Y_{g,n}$, and $\coprod C(X_g \setminus \{*\}; Y)$ are all H -spaces, and the inclusion $Y_{g,n} \rightarrow \text{Map}_*(X_g, S^n)$ and approximation map $C(X_g \setminus \{*\}; Y) \rightarrow \text{Map}_*(X_g, \Sigma^2 Y)$ preserve the product up to homotopy. The product on the configuration model preserves the filtration coming from the number of points in the configuration.*

Notice that restricting the permutation bundle on $C^{i+j}(X_{g+h} \setminus \text{disk})$ to $C^i(X_g \setminus \text{disk}) \times C^j(X_h \setminus \text{disk})$ via κ is precisely the product of permutation bundles:

$$\kappa^*(\gamma_{i+j}) = \gamma_i \times \gamma_j.$$

This is immediate when one thinks of the total space of the permutation bundle over $C^k(X)$ as k points in X labelled by real numbers. Thus $\kappa^*(\gamma_{i+j})$ is pairs of i -tuples of labelled points in $X_g \setminus \text{disk}$ and j -tuples of labelled points in $X_h \setminus \text{disk}$.

Note that $X_0 \setminus \text{disk} = \text{disk}$. Picking an arbitrary point $x_0 \in C^1(X_0 \setminus \text{disk}) = \text{disk}$ and restricting κ to $x_0 \times C^j(X_h \setminus \text{disk})$ gives an alternate construction of the map

$$i_j : C^j(X_h \setminus \text{disk}) \rightarrow C^{j+1}(X_h \setminus \text{disk})$$

from Section 2.1. Consequently, $i_j^*(\gamma_{j+1}) = 1 \oplus \gamma_j$. We conclude by noting that the choice of x_0 does not matter up to homotopy, as $C^1(X_0 \setminus \text{disk})$ is contractible.

4. General computations

This section comprises some preliminary (largely cohomological) computations necessary for the proofs of the main theorems in the next sections. Using the Hopf fibration, we compute the \mathbb{F}_2 -vector space structure of $H^*(\text{Map}_*(X_g, S^2))$ in Section 4.1. This has been computed using different methods in [15, Theorem 4.13]. In Section 4.3 we introduce and compute certain characteristic classes which will allow us in Section 4.4 (along with some facts about the second braid group of the punctured torus) to determine the ring-structure of $H^*(\text{Map}_*(X_1, S^2))$. Knowledge of the case $g = 1$ allows us to finish the computation for $g > 1$ in Section 4.5 using the multiplicative structure described in Section 3.

Throughout this section, we use the isomorphism between $H^*(C^\infty(X_g \setminus \{*\}))$ and $H^*(\text{Map}_*(X_g, S^2)_0)$ (Theorem 2.3). Thus the computations in Section 4.4 and Section 4.5 give the ring structure of the cohomology of the configuration spaces of $X_g \setminus \{*\}$.

4.1. Cohomology of $\text{Map}_*(X_g, S^2)_0$ as a vector space

The Hopf fibration $\eta : S^3 \rightarrow S^2$ gives fibrations $\bar{\eta}$ from mapping spaces induced by postcomposition. Using the fibration $r_n(g) : \text{Map}_*(X_g, S^n) \rightarrow (\Omega S^n)^{\times 2g}$ given by restriction, we get the following commutative diagram of fibrations:

$$\begin{array}{ccccc} \Omega S^1 \simeq * & \longrightarrow & \Omega^2 S^3 & \xrightarrow{\cong/\bar{\eta}} & (\Omega^2 S^2)_0 \\ \downarrow & & \downarrow & & \downarrow \\ \text{Map}_*(X_g, S^1) & \longrightarrow & \text{Map}_*(X_g, S^3) & \xrightarrow{\bar{\eta}} & \text{Map}_*(X_g, S^2)_0 \\ \cong \downarrow & & r_3(g) \downarrow & & r_2(g) \downarrow \\ (\Omega S^1)^{\times 2g} & \longrightarrow & (\Omega S^3)^{\times 2g} & \xrightarrow{\bar{\eta}} & (\Omega S^2)^{\times 2g} \end{array}$$

Looking at the right column, we see that $\bar{\eta}$ defines a map of fibrations from $r_3(g)$ to $r_2(g)$. For the moment, we restrict ourselves to the case $g = 1$. Write

$$H^*((\Omega S^2)^{\times 2}) = \Lambda[x_{2i}, y_{2j}], \quad i, j \geq 0$$

and

$$H^*((\Omega S^3)^{\times 2}) = \Lambda[x_{2i}, y_{2j}], \quad i, j \geq 1$$

where $|x_{2^i}| = 2^i = |y_{2^i}|$. The notation is justified, since $\bar{\eta}^* : H^*(\Omega S^2) \rightarrow H^*(\Omega S^3)$ is an isomorphism in even degrees (and 0 in odd degrees). Therefore $(\bar{\eta} \times \bar{\eta})^* : H^*((\Omega S^2)^{\times 2}) \rightarrow H^*((\Omega S^3)^{\times 2})$ carries x_1 and y_1 to 0, while carrying all other generators x_{2^i}, y_{2^i} to classes of the same name in $H^*((\Omega S^3)^{\times 2})$. For simplicity, if $k = 2^{i_1} + \dots + 2^{i_m}$ is the binary expression for k , write $x_k = x_{2^{i_1}} \cdots x_{2^{i_m}}$. Do similarly for y_k .

Proposition 4.1. *The cohomology Serre spectral sequence for $r_2(1)$ collapses at the E_2 term. On the E_2 terms, the map $\bar{\eta} : \text{Map}_*(X_1, S^3) \rightarrow \text{Map}_*(X_1, S^2)$ induces the homomorphism*

$$H^*((\Omega^2 S^2)_0) \otimes \Lambda[x_{2^i}, y_{2^j}] \rightarrow H^*(\Omega^2 S^3) \otimes \Lambda[x_{2^i}, y_{2^j}]$$

given by the identity on the fibre, and $(\bar{\eta} \times \bar{\eta})^$ (described above) on the base.*

Proof. The restriction of $\bar{\eta}$ to the fibre $\Omega^2 S^3$ of r_2 is an equivalence. It is clear from the discussion above that the map on the E_2 term is as described. Since the spectral sequence for r_3 collapses at the E_2 term, and η^* is onto, the subspace $H^*((\Omega^2 S^2)_0) \otimes \langle x_{2^k} y_{2^l} \rangle$ must persist in the spectral sequence for r_2 . The remaining classes are of the form $z \otimes (x_m y_n)$, where $z \in H^*((\Omega^2 S^2)_0)$ and at least one of m and n is odd. We now show that such elements must also persist.

Consider the spaces $F_1 = X_1/S^1 \times *$ and $F_2 = X_1/* \times S^1$ defined as the quotients of X_1 by collapsing single circles c_1 and c_2 respectively in the 1-skeleton. The quotient map $X_1 \rightarrow F_i$ define maps $f_i : \text{Map}_*(F_i, S^2)_0 \rightarrow \text{Map}_*(X_1, S^2)_0$. Note that F_i is homotopy equivalent to $S^2 \vee S^1$, so $\text{Map}_*(F_i, S^2)_0 \simeq (\Omega^2 S^2)_0 \times \Omega S^2$. Restriction of maps to the one circle left in the 1-skeleton of F_i again provides a fibration $\text{Map}_*(F_i, S^2)_0 \rightarrow \Omega S^2$, but the previous remark shows that it is trivial.

It is obvious that in the map of E_2 terms given by f_1 , the restriction to the fibre is the identity, and that on the cohomology of the base $\Lambda[x_{s_i}, y_{2^j}] \rightarrow \Lambda[y_{2^j}]$, $x_{s_i} \mapsto 0$ and $y_{2^j} \mapsto y_{2^j}$. Since the spectral sequence for F_1 collapses at the E_2 term, and the map f_1^* is a surjection, all of the elements $z \otimes y_n$ persist. Examining F_2 shows that $z \otimes x_m$ also persist. Since this is a spectral sequence of algebras, all elements must therefore persist. So the spectral sequence for $r_2(1)$ collapses at E_2 . \square

We next show that the Serre spectral sequence for $r_2(g)$ collapses for all higher genus using the product described in Proposition 3.1. Iterated multiplication gives an inclusion

$$k^* : \text{Map}_*(X_1, S^2)_0^{\times g} \hookrightarrow \text{Map}_*(X_g, S^2)_0.$$

Note that the two fibrations $r_2(g)$ and the g -fold product $r_2(1)^{\times g}$ have the same base, $(\Omega S^2)^{\times 2g}$. Since k collapses curves not in the 1-skeleton of X_g , k^* is a map of fibrations from $r_2(1)^{\times g}$ to $r_2(g)$.

Corollary 4.2. *The cohomology Serre spectral sequence for $r_2(g)$ collapses at the E_2 term; there is an isomorphism of \mathbb{F}_2 -vector spaces*

$$H^*(\text{Map}_*(X_g, S^2)_0) \cong H^*(\Omega^2 S^3) \otimes H^*((\Omega S^2)^{\times 2g}).$$

Proof. We show that the map that k^* induces on the respective E_2 terms is an injection. Since the spectral sequence for $r_2(1)^{\times g}$ collapses at the E_2 term, so too must the sequence for $r_2(g)$. Since k^* is a map of fibrations over $(\Omega S^2)^{\times 2g}$, we need only show that k^* induces in injection on the cohomology of the fibres. Restricting to the fibre of $r_2(1)^{\times g}$, $(\Omega^2 S^3)^{\times g}$, k^* is precisely the g -fold loop product on $\Omega^2 S^3$. Therefore k^* is simply g -fold comultiplication in $H^*(\Omega^2 S^3)$, which is injective. \square

These arguments give a computation of the multiplicative structure in homology:

Corollary 4.3. *For each n , the homology of*

$$\coprod_{g \geq 0} \text{Map}_*(X_g, S^n)$$

is isomorphic as a ring to

$$H_*(\Omega^2 S^n) \otimes \mathbb{T}(H_*(\Omega S^n \times \Omega S^n))$$

where $\mathbb{T}(V)$ is the free unital associative (tensor) algebra on V .

Proof. First, notice that, for any space Y , the space

$$FY = \coprod_{g \geq 0} Y^{\times g}$$

is the free associative monoid on Y with unit, and that its homology is the free associative algebra on $H^*(Y)$.

As above, the multiplication

$$\mu = k^* : \text{Map}_*(X_g, S^n) \times \text{Map}_*(X_h, S^n) \rightarrow \text{Map}_*(X_{g+h}, S^n)$$

is a map of fibrations over $(\Omega S^n)^{\times 2(g+h)}$. Consequently the map

$$\coprod_{g \geq 0} r_n(g) : \coprod_{g \geq 0} \text{Map}_*(X_g, S^n) \rightarrow \coprod_{g \geq 0} (\Omega S^n \times \Omega S^n)^{\times g} = F(\Omega S^n \times \Omega S^n)$$

is a map of H -spaces. As in the proof of Corollary 4.2, on the fibre μ is loop multiplication in $\Omega^2 S^n$. Since the Serre spectral sequence for $r_n(g)$ collapses, the corollary follows. \square

This sort of result of course holds for any target space Y for which the Serre spectral sequence for $\text{Map}_*(X_g, Y)$ collapses.

Convention 4.4. Notation for the homology and cohomology of $\text{Map}_*(X_g, S^n)$.

As we will be using these vector spaces throughout, we standardize notation here. We have identified the homology and cohomology of $\Omega^2 S^3$ in Theorem 2.7. $H_*(\Omega S^n)$ is a polynomial algebra on a single generator in dimension $n - 1$, so in the case $n = 2$, we may write (as a vector space)

$$H_*(\text{Map}_*(X_g, S^2)_0) \cong \mathbb{F}_2[a_n, n \geq 1] \otimes \mathbb{F}_2[z_1, z_2, \dots, z_{2g-1}, z_{2g}]$$

where $|z_i| = 1$, and z_{2i} and z_{2i-1} come from the i th handle of X_g . For cohomology, $H^*(\Omega S^n)$ is a divided power algebra on a generator in dimension $n - 1$, thus isomorphic to the exterior algebra $\Lambda[x_{2^i}, i \geq 0]$, where the dimension of x_{2^i} is $2^i(n - 1)$. The Steenrod structure is trivial: $Sq(x) = x$, for all x . So we write

$$H^*(\text{Map}_*(X_g, S^2)_0) \cong \Lambda[b_{n,k}] \otimes \Lambda[x_{2^i}^{(j)}, y_{2^i}^{(j)}]$$

where, in the latter term, the superscript (j) identifies the handle from which the term comes.

In the cases $n > 2$, we will only need notation for the homology of $\text{Map}_*(X_g, S^n)$; using Proposition 2.5 we will write

$$H_*(\text{Map}_*(X_g, S^n)) = H_*(\Omega^2 S^n) \otimes \mathbb{F}_2[l_1, l_2, \dots, l_{2g-1}, l_{2g}]$$

where l_i are of dimension $n - 1$. We will also use this notation in the case $n = 2$ if we are making statements about all n simultaneously.

4.2. Homology of $Y_{g,n}$

Recall that $Y_{g,n}$ is the total space of a fibration over $(S^{n-1})^{\times 2g}$ with fiber $\Omega^2 S^n$ ($\Omega^2 S^3$ in the case $n = 2$), defined as the following pullback:

$$\begin{array}{ccc} Y_{g,n} & \longrightarrow & \text{Map}_*(X_g, S^n) \\ \downarrow & & \downarrow r_n(g) \\ (S^{n-1})^{\times 2g} & \xrightarrow{E^{\times 2g}} & (\Omega S^n)^{\times 2g} \end{array}$$

We may identify $H_*((S^{n-1})^{\times 2g}) \subseteq H_*(\Omega S^n)^{\times 2g}$ as the subspace

$$\Lambda_g := \langle l_1^{i_1} \cdots l_{2g}^{i_{2g}} \mid i_n = 0, 1 \rangle \subseteq \mathbb{F}_2[l_1, \dots, l_{2g}]$$

(as a vector space, $\Lambda_g \simeq \Lambda[l_1, \dots, l_g]$).

Proposition 4.5. $H_*(Y_{g,n}) \cong H_*(\Omega^2 S^n) \otimes \Lambda_g$.

Proof. The commutative diagram above shows that $Y_{g,n} \rightarrow \text{Map}_*(X_g, S^n)$ is a map of fibrations, so induces a map on the E_2 terms of the respective (homology) Serre spectral sequences. The E_2 term for $Y_{g,n}$ is $H^*(\Omega^2 S^n) \otimes \Lambda_g$, and the map is the obvious inclusion. Since the spectral sequence for $\text{Map}_*(X_g, S^n)$ collapses at the E_2 -term, so too does the spectral sequence for $Y_{g,n}$. \square

4.3. Characteristic classes for permutation bundles

We now define “conjugate Stiefel–Whitney classes” for bundles, and compute them for the permutation bundles γ_j . Recall that the Stiefel–Whitney classes of a bundle ξ on a space Y with Thom class u are defined by

$$w_i(\xi)u = Sq^i(u).$$

Definition 4.6. Define the conjugate Stiefel–Whitney classes $\bar{w}_i(\xi) \in H^i(Y)$ by the formula

$$\bar{w}_i(\xi)u = \chi(Sq^i)(u).$$

For instance, since $\bar{1} = 1$, $\chi(Sq^1) = Sq^1$, and $\chi(Sq^2) = Sq^2$, $\bar{w}_0 = 1$, $\bar{w}_1 = w_1$, and $\bar{w}_2 = w_2$. Let $\bar{w} = 1 + \bar{w}_1 + \bar{w}_2 + \dots$ denote the “total conjugate Stiefel–Whitney class”.

Since conjugation is an anti-automorphism of \mathcal{A}^* , in principle, $\{\bar{w}_i\}$ carries as much information as $\{w_i\}$. Moreover, they are uniquely suited for bundles whose Thom space is a Brown–Gitler spectrum. The \bar{w}_i enjoy most of the properties of the w_i ; they are natural and satisfy the same Cartan formula for products. In terms of the axiomatic description of Stiefel–Whitney classes given in Chapter 4 of [20], the conjugate Stiefel–Whitney classes satisfy all of the axioms but the requirement that $\bar{w}_n(\xi) = 0$ for $n > \dim(\xi)$. For instance, if γ is the tautological line bundle $\mathbb{R}P^\infty$, $\bar{w}_3(\gamma) = Sq^2 Sq^1 u \neq 0$, where $u \in H^1((\mathbb{R}P^\infty)^\gamma)$ is the Thom class.

Recall that the Snaith splitting of $\Omega^2 S_0^2 \simeq \Omega^2 S^3$ gives

$$H^*(C^\infty(\mathbb{R}^2)) = \bigoplus_{k \geq 0} \Sigma^k M \left[\frac{k}{2} \right] = \bigoplus_{k \geq 0} M \left[\frac{k}{2} \right] \cdot \bar{w}_k$$

where we suggestively write \bar{w}_k for the k -dimensional cyclic generator of $\Sigma^k M \left[\frac{k}{2} \right]$. $H^*(C^j(\mathbb{R}^2))$ is the (split) submodule of terms with $k \leq j/2$. See Theorem 2.7 for an equivalent definition of \bar{w}_k .

Lemma 4.7. The k th conjugate Stiefel–Whitney class of γ_j over $C^j(\mathbb{R}^2)$ is \bar{w}_k for $k \leq j/2$, and 0 otherwise.

Proof. Let u_k be the Thom class of γ_k . Normalize Thom spaces so that their Thom classes are in dimension 0. We include $i_k : C^k(\mathbb{R}^2) \hookrightarrow C^{k+1}(\mathbb{R}^2)$ via the maps i_k defined in Sections 2.1 or 3. Consider the diagram induced by $i_{2k-1} \circ i_{2k-2}$:

$$\begin{array}{ccc} \gamma_{2k-2} \oplus 2 & \longrightarrow & \gamma_{2k} \\ \downarrow & & \downarrow \\ C^{2k-2}(\mathbb{R}^2) & \longrightarrow & C^{2k}(\mathbb{R}^2) \end{array}$$

Upon passage to Thom spaces, this map is $C^{2k-2}(\mathbb{R}^2)^{\gamma_{2k-2}} \rightarrow C^{2k}(\mathbb{R}^2)^{\gamma_{2k}}$. In cohomology this induces the surjection in the short exact sequence of \mathcal{A}^* -modules

$$0 \rightarrow M \left[\frac{k}{2} \right] \cdot \chi(Sq^k) \rightarrow M(k) \rightarrow M(k-1) \rightarrow 0$$

(see [16] for details). Consequently, $\chi(Sq^k)(u_{2k-2}) = 0$, but $\chi(Sq^k)(u_{2k}) \neq 0$. Equivalently, $\bar{w}_k(\gamma_{2k-2}) = 0$ and $\bar{w}_k(\gamma_{2k}) \neq 0$. Therefore

$$\bar{w}_k(\gamma_{2k}) \in H^*(C^{2k}(\mathbb{R}^2)) / H^*(C^{2k-2}(\mathbb{R}^2)) = \Sigma^k M \left[\frac{k}{2} \right].$$

There is only one class of dimension k , the generator of $\Sigma^k M[\frac{k}{2}]$, which is precisely the class that $\bar{w}_k \in H^*(C^\infty(\mathbb{R}^2))$ restricts to. So \bar{w}_k is the k th conjugate Stiefel–Whitney class of γ_{2k} , and therefore for γ_j for any $j \geq 2k$ by naturality. \square

For configurations in positive genus, the approximation $r_j : C^j(X_g \setminus \{*\}) \rightarrow \text{Map}_*(X_g, S^2)$ realizes $H^*(C^j(X_g \setminus \{*\}))$ as a quotient of

$$H^*(\text{Map}_*(X_g, S^2)_0) = H^*(\Omega^2 S^3) \otimes H^*(\Omega S^2)^{\otimes 2g}.$$

See Section 7.1 for an explicit description the kernel of the map.

Corollary 4.8. *The k th conjugate Stiefel–Whitney class of γ_j over $C^j(X_g \setminus \{*\})$ is $\bar{w}_k \otimes 1$ for $k \leq j/2$, and 0 otherwise.*

Proof. Proposition 1.1 implies that as a module over the Steenrod algebra,

$$H^*(\text{Map}_*(X_g, S^3)) \cong H^*(\Omega^2 S^3) \otimes H^*(\Omega S^3)^{\otimes 2g}.$$

The Thom class of γ_j over $C^j(X_g \setminus \{*\})$ lies in this cohomology, since the spectra $C^j(X_g \setminus \{*\})^{\gamma_j}$ are stable summands of the mapping space. Moreover, it restricts to the Thom class of γ_j over $C^j(\mathbb{R}^2)$. Since the conjugate Steenrod operations on the Thom classes are the same for genus 0 and g , the conjugate Stiefel–Whitney classes are the same. \square

4.4. Cohomology of $\text{Map}_*(X_1, S^2)_0$ as a ring

We compute the ring structure of $H^*(\text{Map}_*(X_1, S^2)_0)$ using the computations in the previous section, a fact about the braid group of the punctured torus, and an action of $\text{Map}_*(X_1, S^3)$ on $\text{Map}_*(X_1, S^2)$, via the Hopf fibration.

Theorem 4.9. *There is a ring isomorphism*

$$H^*(\text{Map}_*(X_1, S^2)_0) \cong \Lambda[b_{n,k}; (n, k) \neq (1, 0)] \otimes \Lambda[x_{2i}, y_{2j}] \otimes \mathbb{F}_2[b_{1,0}] / (b_{1,0}^2 = x_1 y_1).$$

In other words, the cohomology of $\text{Map}_(X_1, S^2)_0$ is that of $\Omega^2 S^3 \times \Omega S^2 \times \Omega S^2$ except that $b_{1,0}^2 = \bar{w}_1^2 = x_1 y_1$.*

Proof. We have shown that the spectral sequence for the fibration $r_2(1)$ computing $H^*(\text{Map}_*(X_1, S^2)_0)$ collapses at the term $E_2 = H^*(\Omega^2 S^3) \otimes H^*(\Omega S^2 \times \Omega S^2)$. This a spectral sequence of algebras, so the ring structure on the graded algebra associated to $H^*(\text{Map}_*(X_1, S^2)_0)$ is that of E_2 . E_2 is an exterior algebra, so to determine the actual ring structure of $H^*(\text{Map}_*(X_1, S^2)_0)$ we simply need to determine the squares of all of the ring generators of E_2 .

The fibration $r_2(1)$ makes $H^*(\Omega S^2 \times \Omega S^2)$ a subalgebra of $H^*(\text{Map}_*(X_1, S^2)_0)$, so the squares of all elements coming from $H^*(\Omega S^2 \times \Omega S^2)$ are 0. So we need to only determine the squares of the elements $b_{n,k} \otimes 1$. In the next lemma, we will show that $(\bar{w}_1 \otimes 1)^2 = 1 \otimes x_1 y_1$. We show in Lemma 4.13 that this is the only nonzero square of a generator. Therefore the construction of the isomorphism is immediate. \square

The following computation is crucial to everything that follows.

Lemma 4.10. *The element $\bar{w}_1 \otimes 1 \in H^1(\text{Map}_*(X_1, S^2)_0)$ has $Sq^1(\bar{w}_1 \otimes 1) = (\bar{w}_1 \otimes 1)^2 = 1 \otimes x_1 y_1$.*

Proof. Let $x \in \mathbb{R}^2 \subseteq X_1 \setminus \{*\} \subseteq X_1$. Consider the commutative diagram:

$$\begin{array}{ccccc} \mathbb{R}^2 \setminus \{x\} & \xrightarrow{j} & \tilde{C}^2(\mathbb{R}^2) & \xrightarrow{p_1} & \mathbb{R}^2 \\ \downarrow i & & \downarrow i & & \downarrow i \\ X_1 \setminus \{x\} & \xrightarrow{j} & \tilde{C}^2(X_1) & \xrightarrow{p_1} & X_1 \end{array}$$

Here p_i is the fibration given by projection onto the first factor, and the vertical maps are induced by inclusions $i : \mathbb{R}^2 \subseteq X_1$. Applying π_1 gives a commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{j_*} & \pi_1(\tilde{C}^2(\mathbb{R}^2)) & \xrightarrow{p_{1*}} & 0 \\
 & & \downarrow i_* & \cong & \downarrow i_* & & \downarrow \\
 0 & \longrightarrow & \mathbb{Z} * \mathbb{Z} & \xrightarrow{j_*} & \pi_1(\tilde{C}^2(X_1)) & \xrightarrow{p_{1*}} & \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0
 \end{array}$$

Let d be the generator of $\pi_1(\mathbb{R}^2 \setminus \{x\})$. Then $i_*(d)$ is homotopic to the attaching map for the 2-cell in X_1 , so if $\pi_1(X_1 \setminus \{x\}) = \mathbb{Z} * \mathbb{Z}$ is generated by a and b , then

$$i_*(d) = aba^{-1}b^{-1}.$$

Thinking of these elements as lying in the appropriate pure braid groups via j_* , we have the same equality in $\pi_1(\tilde{C}^2(X_1))$.

Descending to the unlabelled configuration spaces, we have a commutative diagram of extensions:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \pi_1(\tilde{C}^2(\mathbb{R}^2)) & \longrightarrow & \pi_1(C^2(\mathbb{R}^2)) & \longrightarrow & \mathbb{Z}/2 \longrightarrow 0 \\
 & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* \\
 0 & \longrightarrow & \pi_1(\tilde{C}^2(X_1)) & \longrightarrow & \pi_1(C^2(X_1)) & \longrightarrow & \mathbb{Z}/2 \longrightarrow 0
 \end{array}$$

If c is the generator of $\pi_1(C^2(\mathbb{R}^2)) \cong \mathbb{Z}$, then $c^2 = d$. Let $e = i_*(c)$, so that in $\pi_1(C^2(X_1))$ the following equation holds:

$$e^2 = aba^{-1}b^{-1}.$$

For an alternative approach to this sort of result, see Theorem 1.4 in [15].

After abelianization, this implies that in $H_1(C^2(X_1); \mathbb{Z})$, there is a class $h(e)$ coming from $H_1(C^2(\mathbb{R}^2); \mathbb{Z})$ with $2h(e) = 0$ (here h is the Hurewicz map). Since c is the generator of $\pi_1(C^2(\mathbb{R}^2)) \cong \mathbb{Z}$, $h(e)$ is the image in $H_1(C^2(X_1); \mathbb{Z})$ of the generator of $H_1(C^2(\mathbb{R}^2); \mathbb{Z})$. So the reduction mod 2 of $h(e)$ is the class $a_1 \otimes 1$, where, following [3], we can write

$$H_*(C^2(X_1)) \subseteq H_*(\text{Map}(X_1, S^2)_0) = H_*(\Omega^2 S^3) \otimes H_*(\Omega^2 S^2)^{\otimes 2} \otimes H_*(S^2).$$

Since $2h(e) = 0$, we conclude that the dual to $a_1 \otimes 1, \bar{w}_1 \otimes 1$, admits a nontrivial Sq^1 .

Moreover, the same statement holds in $H^*(C^2(X_1 \setminus \{*\}))$. This follows from a 2-local stable equivalence

$$f : C^2(X_1 \setminus \{*\}) \vee S^2 \rightarrow C^2(X_1).$$

The first component of f is induced by the inclusion $X_1 \setminus \{*\} \subseteq X_1$, and the second component is the composite

$$S^2 \xrightarrow{k} \text{Map}(X_1, S^2)_0 \xrightarrow{\pi} C^2(X_1)$$

where k carries a point in S^2 to the constant map at that point, and π is projection onto a stable summand. That f is a 2-local homology isomorphism follows from the computations in [3] of the homology of configuration spaces.

Note that $Sq^1(\bar{w}_1 \otimes 1) = (\bar{w}_1 \otimes 1)^2$ is a square. Examine the cohomologies of the subspaces $\Omega^2 S^3$ and $\text{Map}_*(F_i, S^2)_0$. Since these are \mathbb{F}_2 exterior algebras (see the proof of Proposition 4.1), and every square in such an algebra vanishes, we must have that $Sq^1(\bar{w}_1 \otimes 1)$ vanishes upon pullback to these cohomologies. The class is of dimension 2, so it must lie in the span of

$$\{\bar{w}_2 \otimes 1, \bar{w}_1 \otimes x_1, \bar{w}_1 \otimes y_1, 1 \otimes x_1 y_1, 1 \otimes x_2, 1 \otimes y_2\}.$$

The only element in this span which will vanish in $H^*(\Omega^2 S^3)$ and $H^*(\text{Map}_*(F_i, S^2)_0)$ is $1 \otimes x_1 y_1$. \square

To show that $(b_{n,k} \otimes 1)^2 = 0$ for $(n, k) \neq (1 > 0)$ we will need the following construction: S^3 acts on S^2 since $S^2 = S^3/S^1$ is a homogeneous space of S^3 via the Hopf fibration. This defines an action

$$\lambda : \text{Map}_*(X_g, S^3) \times \text{Map}_*(X_g, S^2)_0 \rightarrow \text{Map}_*(X_g, S^2)_0$$

by pointwise action: $\lambda(f, g)(x) = f(x) \cdot g(x)$. We therefore get a module structure in homology:

Proposition 4.11. Let $a \otimes b \in H_*(\text{Map}_*(X_1, S^3)) = H_*(\Omega^2 S^3) \otimes H_*(\Omega S^3)^{\times 2}$ and $c \otimes d \in H_*(\text{Map}_*(X_1, S^2)_0)$ using the form $E_2 = H_*(\Omega^2 S^3) \otimes H_*(\Omega S^2)^{\times 2}$,

$$\lambda_*((a \otimes b) \otimes (c \otimes d)) = ac \otimes \bar{\eta}_*(b)d$$

where multiplication in the left term is via the loop product μ_2 on $\Omega^2 S^3$, and on the right by the pairwise loop product μ_1 on $\Omega S^2 \times \Omega S^2$.

Proof. The following diagram commutes

$$\begin{CD} \Omega^2 S^3 \times (\Omega^2 S^2)_0 @>\lambda_2>> (\Omega^2 S^2)_0 \\ @VVV @VVV \\ \text{Map}_*(X_g, S^3) \times \text{Map}_*(X_g, S^2)_0 @>\lambda>> \text{Map}_*(X_g, S^2)_0 \\ @Vr_3(1) \times r_2(1)VV @VVr_2(1)V \\ (\Omega S^3)^{\times 2} \times (\Omega S^2)^{\times 2} @>\lambda_1>> (\Omega S^2)^{\times 2} \end{CD}$$

where λ_i are defined as λ is defined with differing domains to the mapping spaces. Therefore computation of the formula above reduces to the computation of λ_{i*} . For the fibre, we have the commutative diagram

$$\begin{CD} \Omega^2 S^3 \times \Omega^2 S^3 @>>> \Omega^2 S^3 \\ @V1 \times \bar{\eta} \simeq VV @VV\bar{\eta} \simeq V \\ \Omega^2 S^3 \times (\Omega^2 S^2)_0 @>\lambda_2>> (\Omega^2 S^2)_0 \end{CD}$$

where the top line is induced by multiplication in S^3 . Since any two H -space products are homotopic, we may as well take it to be the loop product μ_2 . So λ_* behaves as expected on the fibre. The same sort of argument computes the module structure on the base of the fibration. \square

From this proposition and Proposition 4.1, we obtain the following corollary.

Corollary 4.12. $H_*(\text{Map}_*(X_1, S^2)_0)$ is a free module over $H_*(\text{Map}_*(X_1, S^3))$ on generators $\{1, z_1, z_2, z_1 z_2\}$.

We can use this module structure to get at the product in cohomology. Let $\psi : H_* Y \rightarrow H_* Y \otimes H_* Y$ be the comultiplication map. Since λ_* is induced by maps of spaces it is natural with respect to ψ .

Lemma 4.13. For $(n, k) \neq (1, 0)$, $b_{n,k}^2 = 0$.

Proof. This is equivalent to saying that for $(n, k) \neq (1, 0)$, $a_n^{2^k} \otimes a_n^{2^k}$ does not appear in $\psi(x)$ for any element $x \in H_*(\text{Map}_*(X_1, S^2)_0)$. Lemma 4.10 implies that

$$\psi(z_1 z_2) = 1 \otimes z_1 z_2 + z_1 z_2 \otimes 1 + z_1 \otimes z_2 + z_2 \otimes z_1 + a_1 \otimes a_1.$$

Also, by dimensional considerations, $\psi(z_i) = z_i \otimes 1 + 1 \otimes z_i$.

Since $\text{Map}_*(X_1, S^3) \simeq \Omega^2 S^3 \times (\Omega S^3)^{\times 2}$, its cohomology is an exterior algebra. So for any $\alpha \in H_*(\text{Map}_*(X_1, S^3))$, $\psi(\alpha)$ does not include any terms of the form $\beta \otimes \beta$.

We have shown that any $x \in H^*(\text{Map}_*(X_1, S^2)_0)$ may be written as $x = \lambda_*(\alpha \otimes \gamma)$ for some $\gamma \in \langle 1, z_1, z_2, z_1 z_2 \rangle$, so by naturality

$$\psi(x) = \lambda_*(\psi(\alpha) \otimes \psi(\gamma)).$$

The only element of the form $\beta \otimes \beta$ which can appear in $\psi(x)$ is therefore $a_1 \otimes a_1$ (coming from $\gamma = z_1 z_2$). \square

4.5. Cohomology of $\text{Map}_*(X_g, S^2)_0$ as a ring

We will compute the ring structure on $H^*(\text{Map}_*(X_g, S^2)_0)$ using the computations made in the previous sections and the multiplication defined in Proposition 3.1. For notational clarity, we will denote the n th permutation bundle in genus g by $\gamma_n(g)$ since we will be discussing several genera simultaneously. Also, let $\bar{w}_j(g)$ denote the j th conjugate Stiefel–Whitney class of $\gamma_n(g)$ for $n > 2j$ (that this is well defined follows from Corollary 4.8).

Iterated multiplication (Section 3) provides a map

$$k^* : \text{Map}_*(X_1, S^2)_0^{\times g} \rightarrow \text{Map}_*(X_g, S^2)_0$$

which is approximated by configuration models

$$\kappa : C^{j_1}(X_1 \setminus \{*\}) \times \cdots \times C^{j_g}(X_1 \setminus \{*\}) \rightarrow C^{j_1 + \cdots + j_g}(X_g \setminus \{*\}).$$

Recall that the permutation bundle pulls back under κ to products of permutation bundles:

$$\kappa^*(\gamma_{j_1 + \cdots + j_g}(g)) = \gamma_{j_1}(1) \times \cdots \times \gamma_{j_g}(1).$$

Thinking of X_{g+1} as the connected sum $X_g \# X_1$ in $g + 1$ different ways provides $g + 1$ different collapse maps $X_{g+1} \rightarrow X_g$ and hence $g + 1$ distinct inclusions

$$i_j : \text{Map}_*(X_g, S^2)_0 \rightarrow \text{Map}_*(X_{g+1}, S^2)_0.$$

We have configuration models for i_j as well; consider the $g + 1$ maps

$$\iota_j : C^k(X_g \setminus \text{disc}) \rightarrow C^k(X_{g+1} \setminus \text{disc})$$

given by the $g + 1$ inclusions $X_g \setminus \text{disc} \rightarrow X_{g+1} \setminus \text{disc}$. Notice that $\iota_j^* \gamma_k(g + 1) = \gamma_k(g)$. The maps extend to $k = \infty$, and we get the homotopy commutative diagram

$$\begin{array}{ccc} \text{Map}_*(X_g, S^2)_0 & \xrightarrow{i_j} & \text{Map}_*(X_{g+1}, S^2)_0 \\ \uparrow & & \uparrow \\ C^\infty(X_g \setminus \text{disc}) & \xrightarrow{\iota_j} & C^\infty(X_{g+1} \setminus \text{disc}) \end{array}$$

Write $H^*((\Omega S^n)^{\times 2g}) = \Lambda[x_{2i}^{(1)}, y_{2i}^{(1)}, \dots, x_{2i}^{(g)}, y_{2i}^{(g)}; i \geq 0]$ where $|x_n^{(j)}| = n$. The following lemma is immediate from the above discussion.

Lemma 4.14. $i_j^* : H^*(\text{Map}_*(X_{g+1}, S^2)_0) \rightarrow H^*(\text{Map}_*(X_g, S^2)_0)$ is the identity on $H^*(\Omega^2 S^3)$, and carries $H^*(\Omega S^2)^{\otimes 2g+2}$ onto $H^*(\Omega S^2)^{\otimes 2g}$ by mapping $x_m^{(j)}$ and $y_m^{(j)}$ to 0. ι_j^* consequently does identically.

Definition 4.15. Define $\sigma_j \in \Lambda[x_1^{(1)}, y_1^{(1)}, \dots, x_1^{(g)}, y_1^{(g)}] \subseteq H^*(\Omega S^2)^{\otimes 2g}$ for $0 \leq j \leq g$ as

$$\sigma_j = \sum_I x_1^{(i_1)} y_1^{(i_1)} \cdots x_1^{(i_j)} y_1^{(i_j)}$$

where the sum is over sequences $I = (i_1, \dots, i_j)$ where each i_n is a different element of $\{1, 2, \dots, g\}$. σ_j is the j th elementary symmetric exterior polynomial in the variables $(x_1^{(i)} y_1^{(i)})$.

Lemma 4.16. In $H^*(\text{Map}_*(X_g, S^2)_0)$, $\bar{w}_j(g)^2 = 0$ for $j > g$, and $\bar{w}_j(g)^2 = \sigma_j$ for $0 \leq j \leq g$.

Proof. Using the Cartan formula, we see that

$$\kappa^*(\bar{w}_j(g)) = \sum_{j_1 + \cdots + j_g = j} \bar{w}_{j_1}(1) \times \cdots \times \bar{w}_{j_g}(1).$$

We know the ring structure on $H^*(\text{Map}_*(X_1, S^2)_0)$: $\bar{w}_1(1)^2 = x_1^{(1)} y_1^{(1)}$, and the squares of all higher conjugate Stiefel–Whitney classes are 0. Therefore $\kappa^*(\bar{w}_j(g)^2) \neq 0$ if and only if $j \leq g$. κ^* is an injection (see the proof of Corollary 4.2), so $\bar{w}_j(g)^2 \neq 0$ if and only if $j \leq g$.

We will show that $\bar{w}_j(g)^2 = \sigma_j$ by inducting on g ; we have shown this in the case $g = 1$. Assume it for g . Since $\iota_j^* \gamma_k(g+1) = \gamma_k(g)$,

$$\iota_j^*(\bar{w}_n(g+1)^2) = \bar{w}_n(g)^2$$

which, for $n \leq g$, is the n th elementary symmetric exterior polynomial in the g variables $(x_1^{(i)} y_1^{(i)})$, $i \neq j$. Thus

$$\bar{w}_n(g+1)^2 = \sum_j \sigma_n((x_1^{(i)} y_1^{(i)}), \dots, (\widehat{x_1^{(j)} y_1^{(j)}}), \dots, (x_1^{(g+1)} y_1^{(g+1)})) + E$$

where $E \in \cap \ker \iota_j$. By examining F_i (see the proof of Proposition 4.1), E must be divisible by $x_1^{(j)} y_1^{(j)}$ for each j . But then $|E| \geq 2g + 2$, so $E = 0$ if $n \leq g$. So $\bar{w}_n(g+1)^2$ is as described. For $n = g + 1$, the same argument implies that $\bar{w}_{g+1}(g+1)^2 = \sigma_{g+1}$. \square

Theorem 4.17. *Let $d = \lfloor \log_2(g) \rfloor$. $H^*(\text{Map}_*(X_g, S^2)_0)$ is isomorphic to the following ring:*

$$\Lambda[b_{n,k}; (n, k) \notin (1, 0), \dots, (1, d)] \otimes \Lambda[x_{2^i}^{(j)}, y_{2^i}^{(j)}] \otimes \mathbb{F}_2[b_{1,0}, \dots, b_{1,d}] / (b_{1,i}^2 = \sigma_{2^i}).$$

In other words, the cohomology of $\text{Map}_(X_g, S^2)_0$ is that of $\Omega^2 S^3 \times (\Omega S^2)^{\times 2g}$ except that $\bar{w}_j^2 = \sigma_j$ for $0 \leq j \leq g$.*

Proof. The only elements whose squares are in question are $b_{n,k}$. We have shown that $\bar{w}_j(g)^2 = \sigma_j$ for $0 \leq j \leq g$, and since $\bar{w}_j = b_{1,j_1} \cdots b_{1,j_r}$ (where $2^{j_1} + \dots + 2^{j_r}$ is the binary expansion of j), we get, for $0 \leq i \leq d$,

$$b_{1,i}^2 = \sigma_{2^i}.$$

We must now show that the remaining $b_{n,k}^2 = 0$. As in the proof of Lemma 4.13, the result will follow if we show that there is no element x whose coproduct expansion $\psi(x)$ contains $a_n^{2^k} \otimes a_n^{2^k}$ for the remaining n and k . As for genus 1, $H_*(\text{Map}_*(X_g, S^2)_0)$ is a free module over $H_*(\text{Map}_*(X_g, S^3))$ with generators

$$\Lambda_g = \langle z_1^{i_1} \cdots z_{2g}^{i_{2g}} \mid i_n = 0, 1 \rangle.$$

Since

$$z_1^{i_1} \cdots z_{2g}^{i_{2g}} = \kappa_*(z_1^{i_1} z_2^{i_2} \otimes \cdots \otimes z_{2g-1}^{i_{2g-1}} z_{2g}^{i_{2g}}),$$

the coproduct

$$\psi(z_1^{i_1} \cdots z_{2g}^{i_{2g}}) = \kappa_*(\psi(z_1^{i_1} z_2^{i_2}) \otimes \cdots \otimes \psi(z_{2g-1}^{i_{2g-1}} z_{2g}^{i_{2g}})),$$

cannot contain any terms of the form $\beta \otimes \beta$ except for $\beta = a_1^n$, and in that case, $n \leq g$. This result is extended to all of $H_*(\text{Map}_*(X_g, S^2)_0)$ using the Hopf fibration as in Lemma 4.13. \square

5. The second configuration space of the torus

In this section we study the stable homotopy types of the spaces $C^2(X_1 \setminus \{*\})^{n/2}$ for $n = 0, 2, 3$. These computations are fundamental to the proofs of the splittings in the next section. The main results are Lemmas 5.1 and 5.5.

First examine $H_*(C^2(X_1 \setminus \{*\}))$. As a subspace of $H_*(\text{Map}_*(X_1, S^2)_0)$,

$$H_*(C^2(X_1 \setminus \{*\})) = \langle 1, a_1, z_1, z_2, z_1^2, z_1 z_2, z_2^2 \rangle$$

(by $\langle S \rangle$, we mean the \mathbb{F}_2 subspace generated by a set S). This follows from Proposition 7.3 or direct calculation. Here the first term has dimension 0, the next three dimension 1, and the last three dimension 2. Dually, we can write

$$H^*(C^2(X_1 \setminus \{*\})) = \langle 1, \bar{w}_1, x_1, y_1, x_2, x_1 y_1, y_2 \rangle$$

(recall that $\bar{w}_1 = b_{1,0}$).

Lemma 5.1. *2-locally, $C^2(X_1 \setminus \{*\})_+$ splits stably as*

$$C^2(X_1 \setminus \{*\})_+ \simeq S^0 \vee S^1 \vee S^1 \vee S^2 \vee \Sigma M\mathbb{Z}/2.$$

Proof. We begin with the inclusion $C^1(X_1 \setminus \{*\})_+ \rightarrow C^2(X_1 \setminus \{*\})_+$. The domain is simply $X_1 \setminus \{*\}_+ \simeq S^0 \vee S^1 \vee S^1$. In homology, the image of this map is clearly $\langle 1, z_1, z_2 \rangle$.

The stable map

$$\Omega S^3 \times \Omega S^3 \rightarrow \Omega S^3 \times \Omega S^3 \times Y_{g,2} \simeq \text{Map}_*(X_1, S^2)_0 \simeq C^\infty(X_1 \setminus \{*\})$$

may be restricted to $S^2 \vee S^2$ (a stable summand of $\Omega S^3 \times \Omega S^3$) and followed with a projection onto the summand $C^2(X_1 \setminus \{*\})$, giving a map $S^2 \vee S^2 \rightarrow C^2(X_1 \setminus \{*\})$ whose image in homology is $\langle z_1^2, z_2^2 \rangle$.

Finally, we need to account for the $\Sigma M\mathbb{Z}/2$. The inclusion

$$j : S^1 = C^2(\mathbb{R}^2) \rightarrow C^2(X_1 \setminus \{*\})$$

carries the fundamental class $[S^1]$ to the class a_1 , which pairs against a cohomology class \bar{w}_1 which supports a nonzero Sq^1 , according to Lemma 4.10. Therefore, there is a copy of the cohomology of $\Sigma M\mathbb{Z}/2$ in the cohomology of $C^2(X_1 \setminus \{*\})$. Collecting these facts together, we have shown that as a module over \mathcal{A}^* ,

$$H^*(C^2(X_1 \setminus \{*\})_+) \simeq H^*(S^0 \vee S^1 \vee S^1 \vee S^2 \vee S^2 \vee \Sigma M\mathbb{Z}/2). \tag{*}$$

We will show using the Adams spectral sequence that j extends to a map $\Sigma M\mathbb{Z}/2 \rightarrow C^2(X_1 \setminus \{*\})_+$.

To do this, we must show that the composite

$$S^1 \xrightarrow{2} S^1 \xrightarrow{j} C^2(X_1 \setminus \{*\})$$

is 0. Since the element $h_0 \in \text{Ext}_{\mathcal{A}^*}^{1,1}(H^*(S^1), H^*(S^1))$ (representing the map $2 : S^1 \rightarrow S^1$) becomes 0 in

$$\text{Ext}_{\mathcal{A}^*}^{1,1}(H^*(\Sigma M\mathbb{Z}/2), H^*(S^1)),$$

it is carried to 0 in $\text{Ext}_{\mathcal{A}^*}^{1,1}(H^*(C^2(X_1 \setminus \{*\})), H^*(S^1))$ under j . We claim that the image of h_0 is not detected in a higher filtration of the spectral sequence. Therefore the composite $j \circ 2$ is zero, and so j extends to a map $\Sigma M\mathbb{Z}/2 \rightarrow C^2(X_1 \setminus \{*\})$. Moreover, it is clear from Lemma 4.10 that the image of $H_*(\Sigma M\mathbb{Z}/2)$ in $H_*(C^2(X_1 \setminus \{*\}))$ is $\langle a_1, z_1 z_2 \rangle$. Wedging these maps together gives an isomorphism in homology, and thus an equivalence.

Let us examine the possible classes that could detect $j \circ 2$ in the spectral sequence; these lie in

$$\text{Ext}_{\mathcal{A}^*}^{n,n}(H^*(C^2(X_1 \setminus \{*\})_+), H^*(S^1))$$

for $n > 1$. This is identifiable from the cohomological splitting (*): the only nonzero classes come from the $H^*(S^1 \vee S^1)$ component:

$$\text{Ext}_{\mathcal{A}^*}^{n,n}(H^*(S^1 \vee S^1), H^*(S^1)) = \langle h_0^n \rangle \oplus \langle h_0^n \rangle.$$

But we know that $S^1 \vee S^1$ splits off $C^2(X_1 \setminus \{*\})$, so these cannot be targets. \square

Now consider the Thom spaces of multiples of the permutation bundle, $C^2(X_1 \setminus \{*\})^{n\gamma_2}$. We do not study the case $n = 1$, given that we know that $\text{Map}_*(X_1, S^3) \simeq \Omega^2 S^3 \times \Omega S^3 \times \Omega S^3$, so we already have a perfectly good splitting of the Thom spaces of the configuration bundles in this case.

$\tilde{H}_*(C^2(X_1 \setminus \{*\})^{n\gamma_2})$ is \mathbb{F}_2 -isomorphic to the $2n$ -fold suspension of $H_*(C^2(X_1 \setminus \{*\}))$, described above. We would like to split $C^2(X_1 \setminus \{*\})^{n\gamma_2}$ in the same fashion as above, using the Steenrod operations on the cohomology to indicate the pieces it splits into. We will content ourselves with the following partial results. Let $C(\eta + 2)$ be the cofiber of the map $\eta + 2 : S^1 \rightarrow S^0 \vee S^1$.

Lemma 5.2. *$H^*(C^2(X_1 \setminus \{*\})^{2\gamma_2})$ is isomorphic as a module over \mathcal{A}^* to the cohomology of the spectrum*

$$\Sigma^4 C(\eta + 2) \vee S^5 \vee S^5 \vee S^6 \vee S^6.$$

Proof. Let u_2 be the Thom class of $2\gamma_2$. The conjugate Stiefel–Whitney classes of γ_k were determined in Section 4.3. The computation in Theorem 4.9 of the ring structure of $H^*(\text{Map}_*(X_1, S^2)_0)$ (and hence $H^*(C^2(X_1 \setminus \{*\}))$) allows us to determine the conjugate Stiefel–Whitney classes of $2\gamma_2$. Specifically, $\bar{w}(\gamma_2) = 1 + \bar{w}_1$, so

$$\bar{w}(2\gamma_2) = (1 + \bar{w}_1)^2 = 1 + x_1y_1.$$

We can compute conjugate Steenrod operations in $H^*(C^2(X_1 \setminus \{*\})^{2\gamma_2})$ from this information:

$$\chi(Sq)(xu_2) = \chi(Sq)(x)\bar{w}(2\gamma_2)u_2.$$

So specifically,

$$\chi(Sq^n)(xu_2) = (\chi(Sq^n)(x) + \chi(Sq^{n-2})(x)x_1y_1)u_2.$$

The only element of $H^*(C^2(X_1 \setminus \{*\}))$ with any nontrivial Steenrod operations is \bar{w}_1 , with $\chi(Sq^1)\bar{w}_1 = x_1y_1$. Moreover, the only element whose product with x_1y_1 is nonzero is 1. Consequently $\langle x_1u_2 \rangle, \langle y_1u_2 \rangle, \langle x_2u_2 \rangle, \langle y_2u_2 \rangle$, are all split (trivial) \mathcal{A}^* -submodules in dimensions 5 and 6, and $\langle 1u_2, \bar{w}_1u_2, x_1y_1u_2 \rangle$ has the cohomology of $\Sigma^4C(\eta + 2)$. \square

There is a map $\iota : S^0 \rightarrow M\mathbb{Z}/2$ which is the inclusion of the bottom cell. Consider the composite η_1 given by

$$S^1 \xrightarrow{\eta} S^0 \xrightarrow{\iota} M\mathbb{Z}/2$$

and let $C\eta_1$ be its cofiber. The referee has pointed out that $C\eta_1$ is the 2-dual of $C(\eta + 2)$.

Lemma 5.3. $H^*(C^2(X_1 \setminus \{*\})^{3\gamma_2})$ is isomorphic as a module over \mathcal{A}^* to the cohomology of the spectrum

$$\Sigma^6C\eta_1 \vee S^7 \vee S^7 \vee S^8 \vee S^8.$$

The proof is identical to the previous, with the modification that

$$\bar{w}(3\gamma_2) = (1 + \bar{w}_1)^3 = 1 + \bar{w}_1 + x_1y_1.$$

Definition 5.4. Define $g_2 : S^1 \rightarrow C^2(\mathbb{R}^2) = S^1$ to be multiplication by 2. For $n > 2$ and $n \not\equiv 3 \pmod{4}$, define $g_n : S^{2n-3} \rightarrow C^2(\mathbb{R}^2)^{(n-2)\gamma_2}$ as follows: First, set $g_4 : S^5 \rightarrow S^4 \vee S^5$ be $\Sigma^4(\eta + 2)$ and set $g_5 : S^7 \rightarrow \Sigma^6B(1) = \Sigma^6M\mathbb{Z}/2$ to be $\Sigma^6\eta_1$. Let $g_6 : S^9 \rightarrow S^8 \vee S^9$

$$S^9 \xrightarrow{2} S^9 \rightarrow S^8 \vee S^9.$$

Then, for $n > 6$, since $C^2(\mathbb{R}^2)^{(n+2)\gamma_2} \simeq \Sigma^8C^2(\mathbb{R}^2)^{(n-2)\gamma_2}$, set $g_{n+4} = \Sigma^8g_n$.

Lemma 5.5. There are maps

$$Cg_2 \rightarrow D^2(X_1 \setminus \{*\})$$

and for $n > 2$ and $n \not\equiv 3 \pmod{4}$

$$Cg_n \rightarrow C^2(X_1 \setminus \{*\})^{(n-2)\gamma_2}$$

which for $n > 2$ induce an isomorphism of the homology of the domain with the subspace $H_*(C^2(\mathbb{R}^2))^{(n-2)\gamma_2} + \langle l_1l_2 \rangle$. For $n = 2$ the map is an isomorphism onto $H_*(D^2(\mathbb{R}^2)) + \langle l_1l_2 \rangle$. Moreover, after composing these maps with the stable inclusions into $\text{Map}_*(X_1, S^n)$, we may factor them through $Y_{1,n}$.

Proof. The case $n = 2$ is handled by Lemma 5.1. $C^2(\mathbb{R}^2)_+ \simeq S^0 \vee S^1$ so $C^2(\mathbb{R}^2)^{2\gamma_2} \simeq S^4 \vee S^5$. So the map $C^2(\mathbb{R}^2)^{2\gamma_2} \rightarrow C^2(X_1 \setminus \{*\})^{2\gamma_2}$ extends to $\Sigma^4C(\eta + 2)$ for the same reason that $S^1 \rightarrow C^2(X_1 \setminus \{*\})$ extended to $\Sigma M\mathbb{Z}/2$ in Lemma 5.1. That is, the element of $\text{Ext}_{\mathcal{A}^*}^{1,1}(H^*(S^4 \vee S^5), H^*(S^5))$ defining $\eta + 2$ is carried to 0 in $\text{Ext}_{\mathcal{A}^*}^{1,1}(H^*(C^2(X_1 \setminus \{*\})^{2\gamma_2}), H^*(S^5))$ for cohomological reasons. A sparsity argument similar to that of Lemma 5.1 shows that $\eta + 2$ is not detected by an element in a higher filtration of the spectral sequence. Consequently the inclusion $C^2(\mathbb{R}^2)^{2\gamma_2} \rightarrow C^2(X_1 \setminus \{*\})^{2\gamma_2}$ extends to $Cg_4 = \Sigma^4C(\eta + 2)$.

Similarly, $C^2(\mathbb{R}^2)^{3\gamma_2} \simeq \Sigma^6 M\mathbb{Z}/2$ and so $C^2(\mathbb{R}^2)^{3\gamma_2} \rightarrow C^2(X_1 \setminus \{*\})^{3\gamma_2}$ extends to $\Sigma^6 C\eta_1$ in the same fashion. We get the map $Cg_6 \rightarrow C^2(X_1 \setminus \{*\})^{4\gamma_2}$ from $Cg_2 \rightarrow D^2(X_1 \setminus \{*\}) \rightarrow C^2(X_1 \setminus \{*\})$ and the fact that $C^2(X_1 \setminus \{*\})^{4\gamma_2} \simeq \Sigma^8 C^2(X_1 \setminus \{*\})_+$. We extend for $n > 6$, again using the fact that $4\gamma_k$ is trivial for surfaces.

That the image in homology is as described is clear for $n = 2$ from the proof of Lemma 5.1. For $n > 2$, that the map is an injection follows from Lemmas 5.2 and 5.3. The argument to show the image is as described is similar to the proof of Lemma 4.10—the class does not come from $H_*(C^2(\mathbb{R}^2)^{(n-2)\gamma_2})$ and must be symmetric in l_1 and l_2 .

Finally, these maps exist because copies of the cohomologies of Cg_n exist in the cohomology of the configuration models. Examining what survives in

$$H^*(\text{Map}_*(X_1, S^n)) \rightarrow H^*(Y_{1,n})$$

we see that the same modules also exist in $H^*(Y_{1,n})$, so the maps must factor. \square

6. Construction of new spectra and proofs of the main theorems

In this section we construct certain spectra $M_{g,n}(k)$ described in the introduction and demonstrate that they form stable summands of $Y_{g,n+}$ for $n \not\equiv 3 \pmod 4$. Both the construction of these spectra and the proofs of the splittings are inductive in g : we need the splitting of $Y_{g,n+}$ in order to construct the spectra for genus $g + 1$. We wonder if there might be a more formal construction of these spectra which exists without recourse to the geometry of the spaces being split.

The proofs of the splittings of Theorems 1.3 and 1.7 are in two steps. First, we split $Y_{g,n+}$ into a finite wedge of suspensions of spectra we call $M_{h,n}$ (for $h \leq g$). We then split $M_{h,n}$ into a wedge $\bigvee_{k \geq 0} M_{h,n}(k)$.

6.1. Splitting $Y_{g,n+}$

The construction of the spectra $M_{g,n}$ is not dissimilar from the construction of the Smith–Toda complexes when possible. Let us outline the general philosophy. Let R be a (homotopy) commutative, (homotopy) associative ring spectrum with multiplication μ , and let $f : S^n \rightarrow R$ be an element of $\pi_n R$. One may try to construct various R -module spectra R_g for $g \geq 0$ from this data. Let $R_0 = R$. Define R_1 as the cofiber of the map

$$R \wedge S^n \xrightarrow{1 \wedge f} R \wedge R \xrightarrow{\mu} R$$

Since R is associative, the multiplication $\mu : R \wedge R \rightarrow R$ extends to a map $\theta_1 : R \wedge R_1 \rightarrow R_1$ via the diagram

$$\begin{array}{ccccccc} R \wedge R \wedge S^n & \xrightarrow{1 \wedge 1 \wedge f} & R \wedge R \wedge R & \xrightarrow{1 \wedge \mu} & R \wedge R & \longrightarrow & R \wedge R_1 \\ \mu \wedge 1 \downarrow & & \mu \wedge 1 \downarrow & & \downarrow \mu & & \downarrow \theta_1 \\ R \wedge S^n & \xrightarrow{1 \wedge f} & R \wedge R & \xrightarrow{\mu} & R & \longrightarrow & R_1 \end{array}$$

It is not obvious that amongst the possible choices of θ_1 that we can find one that makes R_1 into an associative left R -module. If so, we can construct a spectrum R_2 as the cofiber of

$$R_1 \wedge S^n \xrightarrow{1 \wedge f} R_1 \wedge R \xrightarrow{T} R \wedge R_1 \xrightarrow{\theta_1} R_1$$

The commutativity of R and the associativity of θ_1 induce a map $\theta_2 : R \wedge R_2 \rightarrow R_2$. Again, there may or may not be a choice of such maps which gives an associative module structure on R_2 . One may clearly iterate this process at this point—to construct such R_g for all g requires the verification of associativity at each level.

The ring spectra R that we will employ in constructing certain module spectra $M_{g,n}$ are the suspension spectra of $\Omega^2 S_+^n$ with the loop product (clearly associative and commutative). However, we make this important (but mild) exception: whenever the space $\Omega^2 S^n$ is used in a statement, for $n = 2$, we mean to take the connected component $(\Omega^2 S^2)_0 \simeq \Omega^2 S^3$ instead. In general,

$$\Omega^2 S_+^{2n+1} \simeq \bigvee_{k=0}^{\infty} \Sigma^{(2n-1)k} B \left[\frac{k}{2} \right]$$

and if $n > 1$

$$\Omega^2 S_+^{2n} \simeq \bigvee_{k=0}^{\infty} \Sigma^{(2n-2)k} C^k(\mathbb{R}^2)_+$$

where $C^k(\mathbb{R}^2)_+ \simeq \bigvee_{i=0}^{\lfloor k/2 \rfloor} B[\frac{i}{2}]$.

Define $f_2: S^1 \rightarrow \Omega^2 S^3$ as the composite

$$S^1 \xrightarrow{g_2} D^2(\mathbb{R}^2) \rightarrow \Omega^2 S_+^3$$

and for $n > 2$ and $n \not\equiv 3 \pmod 4$, define $f_n: S^{2n-3} \rightarrow \Omega^2 S^n$ as

$$S^{2n-3} \xrightarrow{g_n} C^2(\mathbb{R}^2)^{(n-2)\gamma_2} \rightarrow \Omega^2 S^n$$

Lemma 6.1. *There exist spectra $M_{g,n}$ (for $n \not\equiv 3 \pmod 4$ and all g) defined in the fashion above using the ring spectrum $M_{0,n} := \Omega^2 S_+^n$ (in the case $n = 2$, we use $M_{0,2} = \Omega^2 S_+^3$) and the element $f_n \in \pi_{2n-3}(M_{0,n})$. They are associative module spectra over $M_{0,n}$. Moreover, 2-locally, $Y_{g,n+}$ splits stably as*

$$Y_{g,n+} \simeq \bigvee_{i=0}^g (\Sigma^{(n-1)i} M_{g-i,n})^{\vee 2^i \binom{g}{i}}$$

and the action of $M_{0,n}$ on $M_{g,n}$ is inherited from the action of $M_{0,n}$ on $Y_{g,n+}$.

Proof. The result is tautologically true in the case $g = 0$, since $\Omega^2 S^n = Y_{0,n}$. So fix n and assume that the result is true for $Y_{h,n}$ for all $h < g$. We construct $M_{g,n}$ as suggested above: let it be the cofiber of the map

$$M_{g-1,n} \wedge S^{2n-3} \xrightarrow{1 \wedge f_n} M_{g-1,n} \wedge M_{0,n} \xrightarrow{\mu} M_{g-1,n}$$

where μ is the module structure map for $M_{g-1,n}$ (which we have assumed to be associative).

To obtain the terms of the splitting involving $M_{h,n}$ for $h < g$ one maps them in from $Y_{g-1,n+}$ in the following fashion. We have g different maps

$$\text{Map}_*(X_1, S^n) \times \text{Map}_*(X_{g-1}, S^n) \rightarrow \text{Map}_*(X_g, S^n)$$

corresponding to the g different ways to collapse $X_g \rightarrow X_1 \vee X_{g-1}$. These maps restrict to maps

$$Y_{1,n+} \wedge Y_{g-1,n+} \rightarrow Y_{g,n+} \tag{*}$$

There is a map $S^{n-1} \vee S^{n-1} \rightarrow Y_{1,n+}$ whose image in homology is $\langle l_1, l_2 \rangle$: the two projections $p_i: X_1 = S^1 \times S^1 \rightarrow S^1$ give maps

$$S^{n-1} \xrightarrow{E} \Omega S^n \xrightarrow{p_i^*} \text{Map}_*(X_1, S^n).$$

Clearly these maps factor through $Y_{1,n}$; the wedge of the two gives the desired map. Then composing this map with the one given by (*) gives a family of g maps

$$(S^{n-1} \vee S^{n-1}) \wedge Y_{g-1,n+} \rightarrow Y_{g,n+}.$$

These maps inject in homology; the i th has image consisting of all multiples of l_{2i-1} or l_{2i} , but not both (i.e., everything is divisible by l_{2i-1} or l_{2i} but not $l_{2i-1}l_{2i}$). Wedging all g maps together defines

$$\phi_g: ((S^{n-1} \vee S^{n-1}) \wedge Y_{g-1,n+})^{\vee g} \rightarrow Y_{g,n+}.$$

This induces an isomorphism of the homology of the domain with the subspace

$$H_*(\Omega^2 S^n) \otimes \{l_1^{i_1} \cdots l_{2g}^{i_{2g}} \mid \exists j \text{ s.t. } i_{2j-1} = 1, i_{2j} = 0, \text{ or } i_{2j-1} = 0, i_{2j} = 1\}.$$

We may rewrite the domain of ϕ_g as $(\Sigma^{n-1} Y_{g-1,n+})^{\vee 2g}$. Employing the lemma for $Y_{g-1,n+}$ and juggling indices appropriately allow us to write this as

$$(\Sigma^{n-1} Y_{g-1,n+})^{\vee 2g} \simeq \bigvee_{i=1}^g (\Sigma^{(n-1)i} M_{g-i,n})^{\vee 2^i \binom{g}{i}}.$$

So to complete the theorem, we need to produce a map $\alpha_g : M_{g,n} \rightarrow Y_{g,n+}$ that induces an isomorphism of $H_*(M_{g,n})$ with the subspace

$$H_*(\Omega^2 S^n) \otimes \Lambda[l_1 \cdot l_2, l_3 \cdot l_4, \dots, l_{2g-1} \cdot l_{2g}]$$

since the direct sum of this space and the image of ϕ_{g*} is $H_*(Y_{g,n+})$. Then the wedge of α_g and ϕ_g is a homology isomorphism, proving the splitting. From this splitting, $M_{g,n}$ inherits an associative $M_{0,n}$ -module structure from $Y_{g,n+}$. This completes the inductive step, as it allows us to define $M_{g+1,n}$.

Let j denote both the inclusion $\Omega^2 S^n_+ \subseteq Y_{1,n+}$ and any of the g inclusions $Y_{g-1,n+} \rightarrow Y_{g,n+}$. Then the following diagram homotopy commutes:

$$\begin{CD} Y_{g-1,n+} \wedge S^{2n-3} @>1 \wedge f_n>> Y_{g-1,n+} \wedge Y_{0,n+} @>\mu>> Y_{g-1,n+} \\ @. @V1 \times jVV @VVjV \\ Y_{g-1,n+} \wedge Y_{1,n+} @>\mu>> Y_{g,n+} \end{CD}$$

We can, of course, replace $Y_{g-1,n+}$ with the split summand $M_{g-1,n}$. Examining the definition of f_n , we can replace it with g_n , allowing us to rewrite the diagram as

$$\begin{CD} M_{g-1,n} \wedge S^{2n-3} @>1 \wedge g_n>> M_{g-1,n} \wedge C^2(\mathbb{R}^4)^{(n-2)\gamma_2} @>\mu>> M_{g-1,n} \\ @. @V1 \wedge jVV @VVjV \\ M_{g-1,n} \wedge Y_{1,n+} @>\mu>> Y_{g,n+} \end{CD}$$

Now, $j \circ g_n = 0$: the content of Lemma 5.5 is that j extends to the cofiber Cg_n of g_n . Since the cofiber of the top row is $M_{g,n}$, the map j on the right side of the diagram extends to a map $M_{g,n} \rightarrow Y_{g,n+}$. The indeterminacy of this collection of maps is the set $[M_{g-1,n} \wedge S^{2n-2}, Y_{g,n+}]$; if we specify an element β_g in this set we fix a particular map $\alpha_g : M_{g,n} \rightarrow Y_{g,n+}$.

To choose this map, note that since $j \circ g_n = 0$, $1 \wedge j$ extends to the cofiber of $1 \wedge g_n$ (which is $M_{g-1,n} \wedge Cg_n$). Again, we have a number of choices for this map, governed by the indeterminacy $[M_{g-1,n} \wedge S^{2n-2}, M_{g-1,n} \wedge Y_{1,n+}]$. However, Lemma 5.5 provides us with specific maps $Cg_n \rightarrow Y_{1,n+}$. Smashing with $M_{g-1,n}$ gives specific extensions of $1 \wedge j$, thus fixing a particular element β'_g of the indeterminacy $[M_{g-1,n} \wedge S^{2n-2}, M_{g-1,n} \wedge Y_{1,n+}]$. Composing with the map $\mu : M_{g-1,n} \wedge Y_{1,n+} \rightarrow Y_{g,n+}$ defines the element β_g , and hence α_g .

To finish, we need to prove that $\alpha_{g,*}$ induces the claimed map in homology. As a graded vector space, $H_*(M_{g,n}) \cong H_*(M_{g-1,n}) + \Sigma^{2n-2} H_*(M_{g-1,n})$. We know that this commutes:

$$\begin{CD} M_{g-1,n} @>\alpha_{g-1}>> Y_{g-1,n+} \\ @VjVV @VVjV \\ M_{g,n} @>\alpha_g>> Y_{g,n+} \end{CD}$$

Inductively, we assume that $\alpha_{g-1,*} : H_*(M_{g-1,n}) \rightarrow H_*(\Omega^2 S^n) \otimes \Lambda[l_1 \cdot l_2, l_3 \cdot l_4, \dots, l_{2g-3} \cdot l_{2g-2}]$, is an isomorphism. So we need to only show that $\alpha_{g,*}$ carries $\Sigma^{2n-2} H_*(M_{g-1,n})$ isomorphically onto

$$H_*(\Omega^2 S^n) \otimes \Lambda[l_1 \cdot l_2, l_3 \cdot l_4, \dots, l_{2g-3} \cdot l_{2g-2}] \cdot l_{2g-1} \cdot l_{2g}.$$

Our choice of α_g implies the commutativity of this diagram:

$$\begin{CD} M_{g-1,n} \wedge Cg_n @>\mu>> M_{g,n} \\ @VVV @VV\alpha_gV \\ M_{g-1,n} \wedge Y_{1,n+} @>\mu>> Y_{g,n+} \end{CD}$$

If $x \in H_{2n-2}(Cg_n)$ is the generator coming from the extension (i.e., the nonzero class not in $H_*(C^2(\mathbb{R}^2)^{(n-2)\gamma_2})$), then the top row carries $H_*(M_{g-1,n}) \cdot x$ onto the subspace $\Sigma^{2n-2} H_*(M_{g-1,n}) \subseteq H_*(M_{g,n})$. By Lemma 5.5, the composite

of the left and bottom rows carries x to $l_{2g-1} \cdot l_{2g}$, so α_{g*} carries $\Sigma^{2n-2} H_*(M_{g-1,n})$ onto $H^*(\Omega^2 S^n) \otimes \Lambda[l_1 \cdot l_2, l_3 \cdot l_4, \dots, l_{2g-3} \cdot l_{2g-2}] \cdot l_{2g-1} \cdot l_{2g}$, as desired. \square

6.2. Splittings of the spectra $M_{g,n}$

The purpose of this section is to show that the spectra $M_{g,n}$ naturally split into a wedge of simpler spectra. Recall that this is true of $M_{0,n}$. Because the multiplication in $M_{0,n}$ respects these wedge gradings, it is reasonable to believe that $M_{1,n}$ —cofibers of maps defined by the multiplication in $M_{0,n}$ —also split into wedges. Another heuristic argument is that overlaying the splitting of $Y_{g,n}$ in the previous section with the Snaith splitting should give a sharper splitting of $Y_{g,n}$. The first approach—exploiting the multiplicative structure—is easier to implement.

We rewrite the stable splittings of $\Omega^2 S_+^n$ as

$$M_{0,n} \simeq \bigvee_{k \geq 0} M_{0,n}(k)$$

with $M_{0,n}(k) = C^k(\mathbb{R}^2)^{(n-2)\gamma_k}$ if $n > 2$. For $n = 2$, set $M_{0,2}(k) = D^k(\mathbb{R}^2)$. Again, $M_{0,2} \simeq \bigvee_{k \geq 0} M_{0,2}(k)$, but there is some redundancy; $M_{0,2}(2k + 1)$ is contractible (and $M_{0,2}(2k) \simeq \Sigma^k B[\lfloor \frac{k}{2} \rfloor]$). Then the loop multiplication on $M_{0,n}$ is given by

$$M_{0,n}(i) \wedge M_{0,n}(j) \rightarrow M_{0,n}(i + j).$$

Proposition 6.2. *There are spectra $M_{g,n}(k)$ defined for $n \geq 2$, $n \not\equiv 3 \pmod{4}$ and all $g, k \geq 0$, so that*

$$M_{g,n} = \bigvee_{k \geq 0} M_{g,n}(k).$$

The composite of $\alpha_g : M_{g,n} \rightarrow Y_{g,n}$ and the inclusion $Y_{g,n} \rightarrow \text{Map}_*(X_g, S^n)$ carries $M_{g,n}(k)$ into the summand $C^k(X_g \setminus \{*\})^{(n-2)\gamma_k}$ if $n > 2$ (if $n = 2$, $M_{g,2}(k)$ maps into $D^k(X_g \setminus \{*\})$). Consequently, the $M_{0,n}$ -module structure on $M_{g,n}$ respects this wedge grading: the action $\mu : M_{0,n} \wedge M_{g,n} \rightarrow M_{g,n}$ is given by a wedge of maps

$$M_{0,n}(i) \wedge M_{g,n}(j) \rightarrow M_{g,n}(i + j).$$

Finally, for $g \geq 1$, $M_{g,n}(k)$ is constructed as the cofiber of the map

$$M_{g-1,n}(k-2) \wedge S^{2n-3} \xrightarrow{1 \wedge g_n} M_{g-1,n}(k-2) \wedge M_{0,n}(2) \xrightarrow{\mu} M_{g-1,n}(k).$$

Proof. As usual, we induct on g . For $g = 0$, the result is already known. So assume the proposition for all $M_{h,n}$ with $h < g$. Then the following diagram commutes:

$$\begin{array}{ccccc} M_{g-1,n} \wedge S^{2n-3} & \xrightarrow{1 \wedge f_n} & M_{g-1,n} \wedge M_{0,n} & \xrightarrow{\mu} & M_{g-1,n} \\ \uparrow = & & \uparrow & & \uparrow = \\ \bigvee_{k \geq 0} M_{g-1,n}(k) \wedge S^{2n-3} & \xrightarrow{1 \wedge g_n} & \bigvee_{k \geq 0} M_{g-1,n}(k) \wedge M_{0,n}(2) & \xrightarrow{\mu} & \bigvee_{k \geq 0} M_{g-1,n}(k) \end{array}$$

The cofiber of the top row is $M_{g,n}$. Since the action $M_{0,n} \wedge M_{g-1,n} \rightarrow M_{g-1,n}$ respects the wedge grading, the bottom row is a sum of the maps defining $M_{g,n}(k)$ in the statement of the proposition. So the cofiber of the bottom (and hence the top) row splits as the wedge $\bigvee_{k \geq 0} M_{g,n}(k)$.

For the remainder of the proof, we treat the cases $n > 2$; the reader can make the obvious modifications for $n = 2$, noting that g_2 maps into the summand $D^2(\mathbb{R}^2) = M_{0,2}(2)$. To show that $M_{g,n}(k)$ maps into $C^k(X_g \setminus \{*\})^{(n-2)\gamma_k}$ is equivalent to saying that the following composite $\alpha_g(k, r)$

$$M_{g,n}(k) \rightarrow M_{g,n} \xrightarrow{\alpha_g} Y_{g,n} \rightarrow \text{Map}_*(X_g, S^n) \rightarrow C^r(X_g \setminus \{*\})^{(n-2)\gamma_r}$$

is 0 if $r \neq k$. By assumption, the restriction of $\alpha_g(k, r)$ to $M_{g-1,n}(k)$ is $\alpha_{g-1}(k, r)$ (composed with the inclusion $C^r(X_{g-1} \setminus \{*\})^{(n-2)\gamma_r} \rightarrow C^r(X_g \setminus \{*\})^{(n-2)\gamma_r}$), and hence zero if $r \neq k$. So $\alpha_g(k, r)$ lifts to $M_{g-1,n}(k-2) \wedge S^{2n-2}$. This lift is a component of the map β_g , so might be denoted $\beta_g(k, r)$. β_g is the indeterminacy of the map

$M_{g-1,n}(k-2) \wedge Cg_n \rightarrow Y_{g,n}$ which extends the map from $M_{g-1,n}(k-2) \wedge C^2(\mathbb{R}^2)^{(n-2)\gamma_2}$ to $Y_{g-1,n}$ given by α_{g-1} and multiplication. Inductively, this last function maps into $C^k(X_{g-1} \setminus \{*\})^{(n-2)\gamma_k}$. So the indeterminacy is 0 if $r \neq k$. Hence $\alpha_g(k, r) = 0$ if $r \neq k$. \square

6.3. Periodicity results for the spectra $M_{g,n}(k)$

In this section, we employ the results of [8]—that $4\gamma_k$ is trivial for surfaces—to prove a similar periodicity for the stable summands of $Y_{g,n}$. As usual, the case $n = 2$ is a slight exception.

Proposition 6.3. For $n \neq 2$,

$$M_{g,n+4}(k) \simeq \Sigma^{4k} M_{g,n}(k).$$

Also,

$$M_{g,6}(k) \simeq \Sigma^{4k} \bigvee_{i=0}^k M_{g,2}(i).$$

Combining the results of this proposition, Lemma 6.1, and Proposition 6.2 gives Theorem 1.7 and hence Theorem 1.3.

Proof. First, take $n \neq 2$. For $g = 0$, this is known— $2\gamma_k$ is trivial over $C^k(\mathbb{R}^2)$ so certainly

$$M_{0,n+4}(k) = C^k(\mathbb{R}^2)^{(n+2)\gamma_k} \simeq \Sigma^{4k} C^k(\mathbb{R}^2)^{(n-2)\gamma_k} = \Sigma^{4k} M_{0,n}(k).$$

Recall that $4\gamma_k$ is trivial over $C^k(X_g \setminus \{*\})$ for each g and k . Thus $C^k(X_g \setminus \{*\})^{(n+2)\gamma_k} \simeq \Sigma^{4k} C^k(X_g \setminus \{*\})^{(n-2)\gamma_k}$. We therefore defined $g_{n+4} = \Sigma^8 g_n$.

Assume the result for $h < g$. Take the sequence whose cofiber is $M_{g,n+4}(k)$,

$$M_{g-1,n+4}(k-2) \wedge S^{2(n+4)-3} \rightarrow M_{g-1,n+4}(k-2) \wedge M_{0,n+4}(2) \rightarrow M_{g-1,n+4}(k), \tag{*}$$

and precompose it with the equivalence $\Sigma^{4k}(M_{g-1,n}(k-2) \wedge S^{2n-3}) \simeq M_{g-1,n+4}(k-2) \wedge S^{2(n+4)-3}$. Postcomposing with $M_{g-1,n+4}(k) \simeq \Sigma^{4k} M_{g-1,n}(k)$ gives a map

$$\Sigma^{4k}(M_{g-1,n}(k-2) \wedge S^{2n-3}) \rightarrow \Sigma^{4k} M_{g-1,n}(k).$$

This is in fact the map whose cofiber is $\Sigma^{4k} M_{g-1,n}(k)$. This is because we have defined $g_{n+4} = \Sigma^8 g_n$ (the first map in $(*)$), and because the configuration model for multiplication (the second map in $(*)$) in dimension $n + 4$ is just a suspension of the model in dimension n . Since we bracketed the defining map for $M_{g,n+4}(k)$ with equivalences to get the defining map for $\Sigma^{4k} M_{g,n}(k)$, these spectra are equivalent.

The difference between the cases $n = 2$ and $n \neq 2$ arises at the base of the induction:

$$M_{0,6}(k) = C^k(\mathbb{R}^2)^{4\gamma_k} \simeq \Sigma^{4k} C^k(\mathbb{R}^2)_+ \simeq \Sigma^{4k} \bigvee_{i=0}^k D^i(\mathbb{R}^2).$$

Since we have defined $D^i(\mathbb{R}^2) = M_{0,2}(i)$, this gives the result for $g = 0$. Again, assume that the result holds for $h < g$; we have the commutative diagram

$$\begin{CD} M_{g-1,6}(k-2) \wedge S^9 @>1 \wedge g_6>> M_{g-1,6}(k-2) \wedge M_{0,6}(2) \\ @V \simeq VV @VV \simeq V \\ \Sigma^{4k}(\bigvee_{i=0}^{k-2} M_{g-1,2}(i) \wedge S^1) @>\Sigma^{4k}(1 \wedge g_2)>> \Sigma^{4k}(\bigvee_{i=0}^{k-2} M_{g-1,2}(i) \wedge C^2(\mathbb{R}^2)_+) \end{CD}$$

In fact g_2 maps into the summand $D^2(\mathbb{R}^2)$. So we may replace $C^2(\mathbb{R}^2)_+$ with $D^2(\mathbb{R}^2) = M_{0,2}(2)$ in the diagram. Then the same inductive argument as in the case $n > 2$ may be used to prove the equivalence for $n = 2$. \square

It is worth noting that $M_{0,4n}(k)$ splits into a wedge as $M_{0,2+4n}(k)$ does. However, the argument in the latter half of the proof above cannot be modified to split $M_{g,4n}(k)$ into even smaller pieces because g_{4n} , unlike g_2 , maps into both summands of $C^2(\mathbb{R}^2)^{(4n-2)\gamma_k}$, whereas g_2 just maps into $M_{0,2}(2)$.

6.4. Characterization of the cohomology of $M_{g,n}(k)$

Write $H_*(M\mathbb{Z}/2) = \langle 1, e \rangle$ and $H_*(C\eta) = \langle 1, f \rangle$ where e and f have dimensions 1 and 2, respectively. The following has as a corollary parts (1) and (2) of Theorem 1.2 via repeated application.

Lemma 6.4.

- (1) $M_{g,2}(2k + 1)$ is contractible.
- (2) There is a map $M_{g,2}(2k) \rightarrow M_{g-1,2}(2k) \wedge M\mathbb{Z}/2$ whose image in homology is

$$H_*(M_{g-1,2}(2k)) \cdot 1 + H_*(M_{g-1,2}(2k - 2)) \cdot e.$$

- (3) There is a map $M_{g,5}(k) \rightarrow M_{g-1,5}(k) \wedge C\eta$ whose image in homology is

$$H_*(M_{g-1,5}(k)) \cdot 1 + H_*(M_{g-1,5}(k - 2)) \cdot f.$$

Proof. For the first item, note that it holds for $g = 0$ and that $M_{g,2}(2k + 1)$ is constructed as the cofiber of a map from $S^1 \wedge M_{g-1,2}(2k - 1)$ to $M_{g-1,2}(2k + 1)$. The result holds by induction.

The following commutes:

$$\begin{array}{ccccc} M_{g-1,2}(2k - 2) \wedge S^1 & \xrightarrow{1 \wedge g_2} & M_{g-1,2}(2k - 2) \wedge M_{0,2}(2) & \xrightarrow{\mu} & M_{g-1,2}(2k) \\ \downarrow = & & & & \downarrow = \\ M_{g-1,2}(2k - 2) \wedge M_{0,2}(2) & \xrightarrow{\mu} & M_{g-1,2}(2k) & \xrightarrow{2} & M_{g-1,2}(2k) \end{array}$$

The cofiber of the top row is $M_{g,2}(2k)$, and the cofiber of the second map on the bottom row is $M_{g-1,2}(2k) \wedge M\mathbb{Z}/2$, thus providing the desired map. That the image in homology is as described is clear, since $H_*(M_{g,2}(2k)) = H_*(M_{g-1,2}(2k)) + \Sigma^2 H_*(M_{g-1,2}(2k - 2))$.

A similar diagram gives the result for $n = 5$:

$$\begin{array}{ccccc} M_{g-1,5}(k - 2) \wedge S^7 & \xrightarrow{1 \wedge g_5} & M_{g-1,5}(k - 2) \wedge M_{0,5}(2) & \xrightarrow{\mu} & M_{g-1,5}(k) \\ \downarrow \Sigma(1 \wedge \iota) & & & & \downarrow = \\ \Sigma M_{g-1,5}(k - 2) \wedge M_{0,5}(2) & \xrightarrow{\Sigma\mu} & \Sigma M_{g-1,5}(k) & \xrightarrow{\eta \wedge 1} & M_{g-1,5}(k) \end{array}$$

where $\iota : S^6 \rightarrow M_{0,5}(2)$ is the inclusion of the bottom cell. \square

We now address part (3) of Theorem 1.2, the computation of the cohomology of $L_g(k) = M_{g,4}(2k)$. Since the stable summands of $\text{Map}_*(X_g, S^4)$ are Thom spaces of bundles over the configuration spaces of $X_g \setminus \{*\}$, they have the same cohomology (as vector spaces) as the configuration spaces, with different Steenrod operations. Our goal is to say the same for the cohomology of $M_{g,4}(k)$ and $\bigvee_{i=0}^k M_{g,2}(i)$. We will use the notation $\alpha_g^{(n)}$ for the map $\alpha_g : M_{g,n}(k) \rightarrow \text{Map}_*(X_g, S^n)$ as we will examine $n = 2$ and $n = 4$ simultaneously.

Lemma 6.5. *The composite*

$$H_*(M_{g,4}(k)) \xrightarrow{\alpha_{g*}^{(4)}} H_*(C^k(X_g \setminus \{*\})^{2\gamma_k}) \xrightarrow{T} H_*(C^k(X_g \setminus \{*\})_+)$$

(where T is the homology Thom isomorphism) carries $H_*(M_{g,4}(k))$ isomorphically onto the image of $H_*(\bigvee_{i=0}^k M_{g,2}(i))$ in $H_*(C^k(X_g \setminus \{*\})_+)$.

Proof. In the case $g = 0$, this holds since $\alpha_0^{(n)}$ is the identity for each n , T is an isomorphism, and

$$C^k(\mathbb{R}^2)_+ \simeq \bigvee_{i=0}^k D^i(\mathbb{R}^2) = \bigvee_{i=0}^k M_{0,2}(i).$$

So assume the result for each $h < g$. Recall that

$$H_*(M_{g,n}(k)) = H_*(M_{g-1,n}(k)) + \Sigma^{2n-2} H_*(M_{g-1,n}(k-2))$$

and that

$$\alpha_{g*}^{(n)}(H_*(M_{g,n}(k))) = \alpha_{g-1*}^{(n)}(H_*(M_{g-1,n}(k))) + \alpha_{g-1*}^{(n)}(H_*(M_{g-1,n}(k-2)))l_{2g-1}l_{2g}.$$

So, by induction

$$T\alpha_{g*}^{(4)}(H_*(M_{g,n}(k))) = \alpha_{g-1*}^{(2)}\left(H_*\left(\bigvee_{i=0}^k M_{g-1,2}(i)\right)\right) + T(\alpha_{g-1*}^{(4)}(H_*(M_{g-1,n}(k-2)))l_{2g-1}l_{2g}).$$

We claim that the last term is $\alpha_{g-1*}^{(2)}(H_*(\bigvee_{i=0}^k M_{g-1,2}(i)))z_{2g-1}z_{2g}$. Let

$$\mu : C^i(X_{g-1} \setminus \{*\})^{2\gamma_i} \wedge C^j(X_1 \setminus \{*\})^{2\gamma_j} \rightarrow C^{i+j}(X_g \setminus \{*\})^{2\gamma_i+j}$$

be the multiplication map. This is induced by a multiplication on the level of configuration spaces; hence the Thom isomorphism respects these multiplicative structures in homology (compare with the proof of Proposition 7.3). We may use μ in our case:

$$T(\alpha_{g-1*}^{(4)}(H_*(M_{g-1,n}(k-2)))l_{2g-1}l_{2g}) = T(\mu_*(\alpha_{g-1*}^{(4)}(H_*(M_{g-1,n}(k-2)))) \otimes l_{2g-1}l_{2g}).$$

It is clear that $T(l_{2g-1}l_{2g}) = z_{2g-1}z_{2g}$, and since multiplication and T commute, this is

$$T(\alpha_{g-1*}^{(4)}(H_*(M_{g-1,n}(k-2))))z_{2g-1}z_{2g}.$$

By induction, this gives the claim. The result follows. \square

We now demonstrate part (3) of Theorem 1.2.

Proof of Theorem 1.2(3). The previous lemma showed that, as graded vector spaces,

$$H_*(M_{g,4}(k)) \cong H_*\left(\bigvee_{i=0}^k M_{g,2}(i)\right)u_k^2$$

where u_k^2 is the (homology) Thom class for $2\gamma_k$. $H_*(M_{g,2}(i)) \subseteq H_*(Y_{g,2})$, so $\alpha_g^{(2)}$ allows us to identify $H_*(M_{g,4}(k))$ as a subset of $H_*(C^k(\mathbb{R}^2)) \otimes \Lambda[z_1z_2, \dots, z_{2g-1}z_{2g}]u_k^2$ (see the proof of Lemma 6.1). So if v_k^2 is the cohomology Thom class, $H^*(M_{g,4}(k))$ is a quotient of

$$H^*(C^k(\mathbb{R}^2)) \otimes \Lambda[x_1^{(1)}y_1^{(1)}, \dots, x_1^{(g)}y_1^{(g)}]v_k^2.$$

The Steenrod operations on this space are given by

$$\chi(Sq^n)(xv_k^2) = \sum_{i=0}^n \chi(Sq^{n-i})(x)\bar{w}_i(2\gamma_k)v_k^2.$$

We have calculated these conjugate Stiefel–Whitney classes:

$$\bar{w}(2\gamma_k) = (\bar{w}(\gamma_k))^2 = 1 + \bar{w}_1^2 + \bar{w}_2^2 + \dots = 1 + \sigma_1 + \sigma_2 + \dots$$

(the σ_i are defined in Section 4.5), so

$$\chi(Sq^n)(xv_k^2) = \sum_{i=0}^g \chi(Sq^{n-2i})(x)\sigma_i v_k^2.$$

Consequently we may write $H^*(M_{g,4}(k))$ as a quotient of

$$\Sigma^{2k} H^* \left(\bigvee_{i=0}^k M_{g,2}(i) \right) \otimes H^*(C\eta)^{\otimes g}$$

where, if we write $H^*(C\eta)^{\otimes g} = \Lambda[t_1, \dots, t_g]$ (with $\dim(t_i) = 2$), t_i is identified with $x_1^{(i)} y_1^{(i)}$, and hence the j th elementary symmetric exterior polynomial in the t_i with σ_j .

This is precisely what part (3) of Theorem 1.2 says. \square

7. Filtration arguments

In this section we construct a weight filtration v on $H_*(C^\infty(X_g \setminus \{*\}))$, the homogeneous summands of which are the homology of the stable summands of $C^\infty(X_g \setminus \{*\})$, $D^k(X_g \setminus \{*\})$. We employ this to give a stable splitting of the configuration spaces and a proof of Proposition 1.4.

7.1. A filtration on $H_*(C^\infty(X_g \setminus \{*\}))$

In [3] a filtration w is put on the homology of $\text{Map}_*(X_g, S^3)$. We may write

$$H_*(\text{Map}_*(X_g, S^3)) \cong H_*(\Omega^2 S^3) \otimes H_*(\Omega S^3)^{\otimes 2g} = \mathbb{F}_2[a_n, n \geq 1] \otimes \mathbb{F}_2[l_1, \dots, l_{2g}]$$

where $|a_n| = 2^n - 1$ and $|l_i| = 2$. Then the filtration w is defined by assigning $w(a_n) = 2^{n-1}$, $w(l_i) = 1$, and extending multiplicatively: $w(ab) = w(a) + w(b)$. In [3], a theorem is proven with the following proposition as a corollary:

Proposition 7.1. [3] *The homology of the k th stable summand of $\text{Map}_*(X_g, S^3)$ given by the Snaitch splitting, $H_*(C^k(X_g \setminus \{*\})^{\gamma_k})$, is the subset of $H_*(\text{Map}_*(X_g, S^3))$ of filtration w equal to k .*

Recall that $H_*(C^\infty(X_g \setminus \{*\})) \cong H_*(\text{Map}_*(X_g, S^2)_0)$. Corollary 4.2 allows us to identify $H_*(C^\infty(X_g \setminus \{*\}))$ as isomorphic to $H_*(\Omega^2 S^3) \otimes H_*(\Omega S^2)^{\otimes 2g}$ as vector spaces. Write $H_*(\Omega S^2)^{\otimes 2g} = \mathbb{F}_2[z_1, \dots, z_{2g}]$ where $|z_i| = 1$.

Definition 7.2. Define a filtration v on

$$H_*(C^\infty(X_g \setminus \{*\})) \cong \mathbb{F}_2[a_n, n \geq 1] \otimes \mathbb{F}_2[z_1, \dots, z_{2g}]$$

by setting $v(a_n) = 2^n$ and $v(z_i) = 1$ and extending via $v(ab) = v(a) + v(b)$.

Notice that if we identify $H_*(\text{Map}_*(X_g, S^3))$ with its image in $H_*(C^\infty(X_g \setminus \{*\})) \cong H_*(\text{Map}_*(X_g, S^2)_0)$ under the Hopf fibration $\tilde{\eta}_*$, a_n is carried to itself, and l_i is carried to z_i^2 . Therefore for $x \in H_*(\text{Map}_*(X_g, S^3))$

$$v(x) = 2w(x).$$

Proposition 7.3. *$H_*(C^k(X_g \setminus \{*\}))$ is the subset of $H_*(C^\infty(X_g \setminus \{*\}))$ of filtration $v \leq k$. Therefore $\tilde{H}_*(D^k(X_g \setminus \{*\}))$ is the subset of filtration precisely k .*

Proof. Clearly, the filtration w restricts to the filtration on $H_*(\Omega^2 S^3)$ defined in Theorem 2.7, whose k th filtered piece is the subspace $A(k) := H_*(C^k(\mathbb{R}^2)^{\gamma_k})$. Therefore the part of $H_*(\text{Map}_*(X_g, S^3))$ of filtration $w = k$ is

$$\tilde{H}_*(C^k(X_g \setminus \{*\})^{\gamma_k}) = \bigoplus_{\Sigma t_i \leq k} A(k - \Sigma t_i) \otimes l_i^{t_1} \cdots l_{2g}^{t_{2g}}.$$

We need to find the subspace of $H_*(C^k(X_g \setminus \{*\}))$ corresponding to this subspace under the Thom isomorphism. Specifically, we have the diagram

$$\begin{array}{ccc} \tilde{H}_*(C^k(X_g \setminus \{*\})^{\gamma_k}) & \xrightarrow{\subseteq} & H_*(\text{Map}_*(X_g, S^3)) \\ \downarrow T_k \cong & & \\ \tilde{H}_*(C^k(X_g \setminus \{*\})) & \xrightarrow{\subseteq} & H_*(C^\infty(X_g \setminus \{*\})) \end{array}$$

where T_k is the (homology) Thom isomorphism for γ_k . We want to compute the image of T_k in $H_*(C^\infty(X_g \setminus \{*\}))$.

We have seen that the module maps $m_{i,j} : C^i(\mathbb{R}^2) \times C^j(X_g \setminus \{*\}) \rightarrow C^{i+j}(X_g \setminus \{*\})$ are covered by maps $\gamma_i \times \gamma_j \rightarrow \gamma_{i+j}$. So the maps induced by $m_{i,j}$ upon passage to Thom spaces

$$\mu_{i,j} : C^i(\mathbb{R}^2)^{\gamma_i} \wedge C^j(X_g \setminus \{*\})^{\gamma_j} \rightarrow C^{i+j}(X_g \setminus \{*\})^{\gamma_{i+j}}$$

commute with the Thom isomorphism. That is,

$$T_{i+j}(\mu_{i,j}(x \otimes y)) = m_{i,j}(T_i(x) \otimes T_j(y)).$$

Briefly, write the products μ and m by a dot \cdot ; it will be apparent from the context which product is being used, and the previous equality shows that it does not matter much anyway.

Pick an element $\alpha \in \tilde{H}_*(C^k(X_g \setminus \{*\})^{\gamma_k})$, and write

$$\alpha = \sum_i \mu_{i,k-i}(a_i u_i(\mathbb{R}^2) \otimes b_i u_{k-i}(X_g \setminus \{*\})) = \sum_i (a_i u_i(\mathbb{R}^2)) \cdot (b_i u_{k-i}(X_g \setminus \{*\}))$$

where $u_i(\mathbb{R}^2)$ is the Thom class of γ_i over $C^i(\mathbb{R}^2)$, and $u_{k-i}(X_g \setminus \{*\})$ is the Thom class of γ_{k-i} over $C^{k-i}(X_g \setminus \{*\})$. Moreover, do this in a fashion in which it is impossible to write $b_i u_{k-i}(X_g \setminus \{*\})$ as a product with some $x u_n(\mathbb{R}^2)$ for some $n > 0$ using the maps μ .

Then $b_i \in H_*(\Omega S^2)^{\otimes 2g}$. By assumption,

$$w((a_i u_i(\mathbb{R}^2)) \cdot (b_i u_{k-i}(X_g \setminus \{*\}))) = k$$

and $w(a_i u_i(\mathbb{R}^2)) = i$, so $w(b_i u_{k-i}(X_g \setminus \{*\})) = k - i$. In general, $w(x) \geq \frac{1}{2}|x|$, so

$$2(k - i) \geq |b_i u_{k-i}(X_g \setminus \{*\})| = |b_i| + k - i$$

and so $v(b_i) = |b_i| \leq k - i$.

We know (using Brown and Peterson’s computation [6] of the homotopy type of the configuration spaces of \mathbb{R}^2 , for instance) that $H_*(C^i(\mathbb{R}^2))$ is the subspace

$$\bigoplus_{n=0}^{\lfloor \frac{i}{2} \rfloor} A(n).$$

These elements are of weight $v \leq i$. So $v(a_i) \leq i$. Therefore $v(a_i \cdot b_i) \leq k$, and so

$$v(T_k(\alpha)) = v\left(\sum_i a_i \cdot b_i\right) \leq k.$$

Consequently T_k maps $\tilde{H}_*(C^k(X_g \setminus \{*\})^{\gamma_k})$ injectively into the space $v \leq k$; by dimension counts, this is a surjection as well. \square

7.2. Splitting configuration spaces

Via McDuff’s work in [18] we obtain the following as a corollary to Theorem 1.3:

Corollary 7.4. *The r th configuration space, $C^r(X_g \setminus \{*\})_+$ of the punctured surface $X_g \setminus \{*\}$ is stably equivalent to the wedge of*

$$(\Sigma^i M_{g-i,2}(k))^{\vee 2^i \binom{g}{i}} \wedge S^{2(j_1 + \dots + j_{2g})}$$

over all nonnegative choices of i, k , and j_1, \dots, j_{2g} so that $i + k + 2(j_1 + \dots + j_{2g}) \leq r$.

Proof. Since $C^k(X_g \setminus \{*\})_+$ is a stable summand of $\text{Map}_*(X_g, S^2)_{0+}$, it is equal to a wedge of a subcollection of the summands of

$$\text{Map}_*(X_g, S^2)_{0+} \simeq \left(\bigvee_{i=0}^g \left(\bigvee_{k=0}^\infty M_{g-i,2}(k) \right)^{\vee 2^i \binom{g}{i}} \right) \wedge \left(\bigvee_{j=0}^\infty S^{2j} \right)^{\wedge 2g}. \tag{**}$$

We determine which subcollection by examining the homology of these summands.

First, we note that $H_*(M_{g,2}(k)) \subseteq H_*(\text{Map}_*(X_g, S^2)_0)$ is homogeneous of fixed weight k , since we know that $M_{g,2}(k)$ maps into $D^k(X_g \setminus \{*\})$ via α_g . Secondly, any of the terms $H_*(S^{2j})$ coming from one of the copies of $H_*(\Omega S^3)$ are of weight $2j$. Similarly, any of the terms $H_*(\Sigma^i M_{g-i,2}(k))$ is therefore of weight $i + k$. Since the definition of v is multiplicative, we obtain all of the terms listed in the corollary. \square

7.3. A proof of Proposition 1.4

Recall the Hopf–James map

$$\Omega S^n \xrightarrow{H} \Omega S^{2n-1}$$

whose fibre, at the prime 2, is given by the suspension map $E : S^{n-1} \rightarrow \Omega S^n$. This leads to the defining diagram of $Y_{g,n}$:

$$\begin{array}{ccccc} Y_{g,n} & \xrightarrow{i} & \text{Map}_*(X_g, S^n) & & \\ \downarrow & & \downarrow r_n(g) & & \\ (S^{n-1})^{\times 2g} & \xrightarrow{E^{\times 2g}} & (\Omega S^n)^{\times 2g} & \xrightarrow{H^{\times 2g}} & (\Omega S^{2n-1})^{\times 2g} \end{array}$$

The composite $(H^{\times 2g} \circ r_n(g))^*$ makes $H^*(\text{Map}_*(X_g, S^n))$ a $H^*((\Omega S^{2n-1})^{\times 2g})$ -module. We will define a subspace $Z_{g,n} \subseteq H^*(\text{Map}_*(X_g, S^n))$ and show that it is a free basis for the action of $H^*((\Omega S^{2n-1})^{\times 2g})$. The proof of Proposition 1.4 will then essentially follow by showing that i_* carries $Z_{g,n}$ isomorphically onto $H^*(Y_{g,n})$.

Recall that $\text{Map}_*(X_g, S^2)_0 \simeq Y_{g,2} \times (\Omega S^3)^{\times 2g}$, and that r_k denotes the approximation map

$$r_k : C^k(X_g \setminus \{*\}) \rightarrow \text{Map}_*(X_g, S^2)_0.$$

Definition 7.5. Define $Z_{g,n} \subseteq H^*(\text{Map}_*(X_g, S^n))$ as the span of the set

$$\{r_k^*(x)u_k^{n-2}, \forall k, \forall x \in H^*(Y_{g,2}) \subseteq H^*(\text{Map}_*(X_g, S^2)_0)\}$$

where u_k^{n-2} is the Thom class of the bundle γ_k^{n-2} over $C^k(X_g \setminus \{*\})$.

In other words, $Z_{g,n}$ is the part of $H^*(\text{Map}_*(X_g, S^2)_0)$ coming from $Y_{g,2}$ lifted to $H^*(\text{Map}_*(X_g, S^n))$ via the various Thom isomorphisms from the stable summands $\text{Map}_*(X_g, S^n)$.

Lemma 7.6. $Z_{g,n}$ is an \mathcal{A}^* -submodule of $H^*(\text{Map}_*(X_g, S^n))$.

Proof. We compute the total conjugate Steenrod operation on a generator using the definition of \bar{w} and naturality with respect to r_k :

$$\chi(Sq)(r_k^*(x)u_k^{n-2}) = (r_k^*(\chi(Sq)(x)))\bar{w}((n-2)\gamma_k)u_k^{n-2}.$$

Bödighheimer, Cohen, and Milgram have shown that $Y_{g,2}$ is a factor of $\text{Map}_*(X_g, S^2)_0$; hence $H^*(Y_{g,2})$ is a sub \mathcal{A}^* -algebra of $H^*(\text{Map}_*(X_g, S^2)_0)$. So since $x \in H^*(Y_{g,2})$

$$\chi(Sq)(x) \in H^*(Y_{g,2}).$$

Moreover, Corollary 4.8 implies that $\bar{w}((n-2)\gamma_k) \in r_k^*(H^*(Y_{g,2}))$. Since $H^*(Y_{g,2})$ is closed under multiplication, this implies that

$$(r_k^*(\chi(Sq)(x)))\bar{w}((n-2)\gamma_k) \in r_k^*(H^*(Y_{g,2}))$$

and so

$$\chi(Sq)(r_k^*(x)u_k^{n-2}) \in Z_{g,n}.$$

Since \mathcal{A}^* is generated by the conjugate Steenrod operations, this demonstrates that $Z_{g,n}$ is closed under the action of the Steenrod algebra. \square

Denote by T_k the Thom isomorphism

$$T_k : \tilde{H}_*(C^k(X_g \setminus \{*\}))^{(n-2)\gamma_k} \rightarrow \tilde{H}_*(C^k(X_g \setminus \{*\})).$$

We need the following computational lemma.

Lemma 7.7. For $a \otimes l_1^{\alpha_1} \cdots l_{2g}^{\alpha_{2g}} \in H_*(Y_{g,n}) \subseteq H_*(\text{Map}_*(X_g, S^n))$ (hence $0 \leq \alpha_i \leq 1$) of filtration $k + \sum \alpha_i$,

$$T_{k+\sum \alpha_i}(a \otimes l_1^{\alpha_1} \cdots l_{2g}^{\alpha_{2g}}) = T_k(a) \otimes (z_1^{\alpha_1} \cdots z_{2g}^{\alpha_{2g}} + HO)$$

where HO are terms involving powers of z_i greater than 1.

Proof. We begin with the case $a = 1, g = 1$; that is,

$$T_{\alpha_1+\alpha_2}(l_1^{\alpha_1} l_2^{\alpha_2}) = z_1^{\alpha_1} z_2^{\alpha_2} + HO.$$

If only one of the α_i are nonzero, this follows from the fact that

$$C_1(X_1 \setminus \{*\}) \simeq S^1 \vee S^1$$

so there is no choice but to have $T_1(l_i) = z_i$.

Next examine $T_2(l_1 l_2)$. The only possible targets lie in

$$H_2(C^2(X_1 \setminus \{*\})) = \langle z_1^2, z_1 z_2, z_2^2 \rangle$$

$\mathbb{Z}/2$ acts on $X_1 = \mathbb{R}^2/\mathbb{Z}^2$ by switching coordinates; this interchanges the two loops in the 1-skeleton. By symmetry with respect to this action, in fact

$$T_2(l_1 l_2) \in \langle z_1^2 + z_2^2, z_1 z_2 \rangle.$$

The subspace $\langle l_1^2, l_2^2 \rangle$ is free $\mathbb{Z}/2$ -module, hence $\langle T_2(l_1^2), T_2(l_2^2) \rangle$ must be as well. The only free subspaces of $H_2(C^2(X_1 \setminus \{*\}))$ are

$$\langle z_1^2, z_2^2 \rangle \quad \text{and} \quad \langle z_1^2 + z_1 z_2, z_2^2 + z_1 z_2 \rangle.$$

Thus $T_2(l_1 l_2)$ must lie in the complement of one of these two subspaces. In either case, this implies that

$$T_2(l_1 l_2) = z_1 z_2 + r(z_1^2 + z_2^2)$$

for some $r \in \mathbb{F}_2$.

The general case now follows from the multiplicative properties used throughout this article. In particular, we use the multiplication

$$C^k(\mathbb{R}^2) \times C^{\alpha_1+\alpha_2}(X_1 \setminus \{*\}) \times \cdots \times C^{\alpha_{2g-1}+\alpha_{2g}}(X_1 \setminus \{*\}) \rightarrow C^{k+\sum \alpha_i}(X_g \setminus \{*\})$$

and the induced map on Thom spaces to obtain the class $a \otimes l_1^{\alpha_1} \cdots l_{2g}^{\alpha_{2g}}$. The result then follows from the fact that multiplication commutes with the Thom isomorphism. \square

Corollary 7.8. $Z_{g,n}$ is carried isomorphically onto $H^*(Y_{g,n})$ via i^* .

Notice that, since i is a map of spaces, this isomorphism is one of \mathcal{A}^* -modules.

Proof. An equivalent formulation of this fact is the statement that the Kronecker product

$$\langle \cdot, \cdot \rangle : H_*(\text{Map}_*(X_g, S^n)) \otimes H^*(\text{Map}_*(X_g, S^n)) \rightarrow \mathbb{F}_2$$

remains nonsingular when restricted to $i_*(H_*(Y_{g,n})) \otimes Z_{g,n}$. A typical generator of $i_*(H_*(Y_{g,n}))$ is of the form

$$A = a \otimes l_1^{\alpha_1} \cdots l_{2g}^{\alpha_{2g}}$$

where $a \in H_*(\Omega^2 S^n)$, $0 \leq \alpha_i \leq 1$. A typical generator of $Z_{g,n}$ is of the form

$$B = r_k^*(b \otimes x^\beta) u_{k+\sum \beta_i}^{n-2}$$

where $b \in H^*(\Omega^2 S_0^2)$, $\beta = (\beta_1, \dots, \beta_{2g})$ is a sequence of $\{0, 1\}$, and

$$x^\beta = (x_1^{(1)})^{\beta_1} (y_1^{(1)})^{\beta_2} \dots (x_1^{(g)})^{\beta_{2g-1}} (y_1^{(g)})^{\beta_{2g}}.$$

Then

$$\begin{aligned} \langle A, B \rangle &= \langle T_{k+\sum \beta_i}(a \otimes l_1^{\alpha_1} \dots l_{2g}^{\alpha_{2g}}), r_{k+\sum \beta_i}^*(b \otimes x^\beta) \rangle \\ &= \langle T_k(a) \otimes (z_1^{\alpha_1} \dots z_{2g}^{\alpha_{2g}} + HO), r_{k+\sum \beta_i}^*(b \otimes x^\beta) \rangle \\ &= \langle T_k(a), r_k^*(b) \rangle \cdot \langle z_1^{\alpha_1} \dots z_{2g}^{\alpha_{2g}} + HO, x^\beta \rangle \\ &= \langle a, r_k^*(b) u_k^{n-2} \rangle \cdot \prod_{i=1}^g \delta_{\alpha_i}^{\beta_i}. \end{aligned} \tag{†}$$

The first term in the last line of (†) is the Kronecker product on $\Omega^2 S^n$. This form is nonsingular, as is the form given by the second term. So (†) implies that the Kronecker product on $i_*(H_*(Y_{g,n})) \otimes Z_{g,n}$ is the tensor product of two nonsingular forms, and is thus also nonsingular. \square

Proof of Proposition 1.4. We claim that $H^*(\text{Map}_*(X_g, S^n))$ is a free $H^*(\Omega S^{2n-1})^{\otimes 2g}$ -module on the subspace $Z_{g,n}$; that is, multiplication in $H^*(\text{Map}_*(X_g, S^n))$ induces an isomorphism

$$H^*(\Omega S^{2n-1})^{\otimes 2g} \otimes Z_{g,n} \rightarrow H^*(\text{Map}_*(X_g, S^n)).$$

Since this isomorphism is induced by multiplication, it is necessarily an isomorphism of \mathcal{A}^* -modules. Then Proposition 1.4 follows from Corollary 7.8. Consider the cohomological Serre spectral sequence of the fibration

$$Y_{g,n} \xrightarrow{i} \text{Map}_*(X_g, S^n) \xrightarrow{j} (\Omega S^{2n-1})^{\times 2g}$$

where $j = H^{\times 2g} \circ r_n(g)$. The E_2 term is of the form

$$E_2 = H^*(\Omega S^{2n-1})^{\otimes 2g} \otimes H^*(Y_{g,n}).$$

The spectral sequence must collapse at E_2 , since (as vector spaces)

$$E_2 \cong H^*(\Omega S^{2n-1})^{\otimes 2g} \otimes H^*(S^{n-1})^{\otimes 2g} \otimes H^*(\Omega^2 S^n) \cong H^*(\Omega S^n)^{\otimes 2g} \otimes H^*(\Omega^2 S^n)$$

which we know to be isomorphic to $E_\infty = H^*(\text{Map}_*(X_g, S^n))$. More carefully, since E_2 is locally finite dimensional, any nonzero differential would reduce the dimension of each graded piece of E_∞ to less than that of E_2 . But since E_2 is abstractly isomorphic to $H^*(\text{Map}_*(X_g, S^n)) \cong E_\infty$ this is not possible.

Restricting the multiplication in $H^*(\text{Map}_*(X_g, S^n)) \otimes H^*(\text{Map}_*(X_g, S^n))$ to the subspace $H^*(\Omega S^{2n-1})^{\otimes 2g} \otimes Z_{g,n}$ gives a map

$$H^*(\Omega S^{2n-1})^{\otimes 2g} \otimes Z_{g,n} \rightarrow H^*(\text{Map}_*(X_g, S^n)).$$

Reducing this multiplication to the filtration quotients from the spectral sequence gives

$$H^*(\Omega S^{2n-1})^{\otimes 2g} \otimes H^*(Y_{g,n}) \rightarrow E_2 \cong E_\infty$$

since, by Corollary 7.8, $H^*(Y_{g,n}) \cong Z_{g,n}$. The spectral sequence is one of algebras; thus this multiplication is an isomorphism. Since the filtered multiplication is an isomorphism, so too is the original multiplication. \square

8. Unbased mapping spaces

We compute the cohomology of the unbased mapping spaces $\text{Map}(X_g, S^n)$. The main tool is the Serre spectral sequence for the fibration

$$\text{Map}(X_g, S^n) \xrightarrow{\text{ev}} S^n$$

given by evaluation of functions at the basepoint of X_g . The fibre is $\text{Map}_*(X_g, S^n)$. We first treat the special case $g = 1$ and $n = 2$:

Lemma 8.1. *The Serre spectral sequence for $\text{Map}(X_1, S^2)_d$ collapses at the E_2 term for maps of every degree d .*

The proof of this follows an approach to these sort of collapsing results described in [14]: one uses the suspension $E : S^2 \rightarrow \Omega S^3$ to map the homology spectral sequence under consideration injectively to one that collapses. For a careful exposition of this proof for $g = 0$, see [15, Lemma 2.5].

Proposition 8.2. *For $n \geq 2$ and $g \geq 0$, the cohomology Serre spectral sequence for the fibration ev collapses at the E_2 term; there is an isomorphism of vector spaces*

$$H^*(\text{Map}(X_g, S^n)) \cong H^*(\text{Map}_*(X_g, S^n)) \otimes H^*(S^n).$$

Proof. For genus 1 and $n > 2$, since X_1 is parallelizable, this follows (like Proposition 2.5) from the main result of [3]; if $n = 2$ we use the previous lemma. For genus 0, consider the commutative diagram of fibrations over S^n :

$$\begin{array}{ccc} \Omega^2 S^n & \longrightarrow & \text{Map}_*(X_1, S^n) \\ \downarrow & & \downarrow \\ \text{Map}(X_0, S^n) & \longrightarrow & \text{Map}(X_1, S^n) \\ \text{ev} \downarrow & & \downarrow \text{ev} \\ S^n & \xrightarrow{=} & S^n \end{array}$$

Since the Serre spectral sequence collapses for the right-hand fibration, and the induced map on the cohomology of the fibres is a surjection, the spectral sequence for the left-hand fibration collapses.

For genus $g > 1$ we have a two step process to proving the collapse of the spectral sequence. Consider the commutative diagram

$$\begin{array}{ccccc} \text{Map}_*(X_1, S^n)^{\times g} & \xleftarrow{=} & \text{Map}_*(X_1, S^n)^{\times g} & \xrightarrow{\mu} & \text{Map}_*(X_g, S^n) \\ \downarrow & & \downarrow & & \downarrow \\ \text{Map}(X_1, S^n)^{\times g} & \xleftarrow{\quad} & \text{Map}(X_1^{\vee g}, S^n) & \xrightarrow{\mu} & \text{Map}(X_g, S^n) \\ \text{ev}^{\times g} \downarrow & & \downarrow \text{ev} & & \downarrow \text{ev} \\ (S^n)^{\times g} & \xleftarrow{\Delta} & S^n & \xrightarrow{=} & S^n \end{array}$$

Here Δ is the g -fold diagonal, and the center fibration is a pullback of the left one over Δ . The map from the center fibration to the right one is induced by the iterated multiplication described in Section 3 and its obvious extension to $\text{Map}(X_1^{\vee g}, S^n)$.

Since the spectral sequence for the left fibration collapses and Δ^* is a surjection, the spectral sequence for the center fibration collapses. Since the map μ on the fibres is an injection in cohomology (as in Corollary 4.2), the spectral sequence for the right-hand fibration collapses. \square

The arguments of Proposition 2.6 can be modified to show that $\text{Map}(X_g, S^3)$ is equivalent to $\text{Map}_*(X_g, S^3) \times S^3$. Despite the evidence of the previous proposition, the analogue for target spheres of arbitrary dimension is not true (even stably) in general:

Proposition 8.3. *If n is even, then $\text{Map}(X_g, S^n)$ is not stably equivalent to $\text{Map}_*(X_g, S^n) \times S^n$.*

Proof. S^1 is a factor of X_1 ; this induces a retraction $\text{Map}(X_1, S^n) \rightarrow LS^n$ (the free loop space of S^n). So LS^n is a stable summand of $\text{Map}(X_1, S^n)$. Were $\text{Map}(X_1, S^n)$ equivalent to $\text{Map}_*(X_1, S^n) \times S^n$, this would give a stable splitting $LS^n \simeq \Omega S^n \times S^n$. While this is true for n odd, it is false for n even. Briefly, LS^n admits a configuration model, and hence a Snaith splitting. The second term is

$$C^2(S^1)_+ \wedge_{\Sigma_2} (S^{n-1} \wedge S^{n-1}) \simeq S^1_+ \wedge_{\Sigma_2} (S^{n-1} \wedge S^{n-1}).$$

If n is even, this is $\Sigma^{2n-2} M\mathbb{Z}/2$, whereas $\Omega S^n \times S^n$ is a wedge of spheres. One may propagate this splitting failure to higher genus using the multiplicative techniques described in previous sections. \square

Acknowledgements

I would like to thank Fred Cohen, Paul Goerss, Igor Kriz, and Mark Mahowald for their helpful conversations on this material. Their insight has been much appreciated. I would also like to thank the referee for many helpful suggestions which improved the exposition of the paper.

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